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Cohomology of Vector Bundles and Syzygies

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2003

Printed in the United States of America

Typeface Times 10/13 pt. *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data

Weyman, Jerzy, 1955–

Cohomology of vector bundles and syzgies / Jerzy Weyman.

p. cm. – (Cambridge tracts in mathematics; 149)

Includes bibliographical references and index.

ISBN 0-521-62197-6

1. Syzygies (Mathematics) 2. Vector bundles. 3. Homology theory. I. Title. II. Series.

QA247 .W49 2003
512′.5 – dc21 2002074071

ISBN 0 521 62197 6 hardback

Contents

1

Introductory Material

1.1. Multilinear Algebra and Combinatorics

1.1.1. Exterior, Divided, and Symmetric Powers; Multiplication and Diagonal Maps

Let **K** be a commutative ring, and let E be a free **K**-module with a basis $\{e_1, \ldots, e_n\}$.

We define *the r-th exterior power $\bigwedge^r E$ of E* to be the r-th tensor power $E^{\otimes r}$ of E divided by the submodule generated by the elements:

$$u_1 \otimes \ldots \otimes u_r - (-1)^{\operatorname{sgn} \sigma} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(r)}$$

for all $\sigma \in \Sigma_r$, $u_1, \ldots, u_r \in E$. We denote the coset of $u_1 \otimes \ldots \otimes u_r$ by $u_1 \wedge \ldots \wedge u_r$.

The following basic properties of exterior powers are proved in [L, chapter XIX, section 1].

(1.1.1) Proposition.

*(a) Let $\{e_1, \ldots, e_n\}$ be an ordered basis of E. Then the elements $e_{i_1} \wedge \ldots \wedge e_{i_r}$ for $1 \leq i_1 < \ldots < i_r \leq n$ form a basis of $\bigwedge^r E$. In particular, $\bigwedge^r E$ is a free **K**-module of dimension $\binom{n}{r}$.*

(b) (Universality property of exterior powers) We have a functorial isomorphism

$$\theta_M : \operatorname{Alt}^r(E^r, M) \to \operatorname{Hom}_{\mathbf{K}}\left(\bigwedge^r E, M\right)$$

where $\operatorname{Alt}^r(E^r, M)$ denotes the set of multilinear alternating maps from $E^{\times r}$ to M, given by the formula $\theta^r_M(f)(u_1 \wedge \ldots \wedge u_r) = f(u_1, \ldots, u_r)$.

(c) We have natural isomorphisms

$$\alpha^r : \bigwedge^r (E^*) \to \left(\bigwedge^r E\right)^*$$

sending the exterior product $l_1 \wedge \ldots \wedge l_r$ *to the linear function* l *on* $\bigwedge^e E$ *defined by the formula*

$$l(u_1 \wedge \ldots \wedge u_r) = \sum_{\sigma \in \Sigma^r} (-1)^{\mathrm{sgn}\,\sigma} l_{\sigma(1)}(u_1) \ldots l_{\sigma(r)}(u_r).$$

The r-th exterior power is an endofunctor on the category of free **K**-modules and linear maps. More precisely, for two free **K**-modules E, F and a linear map $\phi : E \to F$ we have a well-defined linear map

$$\bigwedge^r \phi : \bigwedge^r E \to \bigwedge^r F$$

defined by the formula $\bigwedge^r \phi(u_1 \wedge \ldots \wedge u_r) = \phi(u_1) \wedge \ldots \wedge \phi(u_r)$. Let us denote $m = \dim F$. Let $\{e_1, \ldots, e_n\}$ be a basis of E and let $\{f_1, \ldots, f_m\}$ be a basis of F. In these bases ϕ correspond to the $m \times n$ matrix $(\phi_{j,i})$ where

$$\phi(e_i) = \sum_{j=1}^{m} \phi_{j,i} f_j.$$

The map $\bigwedge^r \phi$ can be written in the corresponding bases of $\bigwedge^r E$, $\bigwedge^r F$ as follows:

$$\bigwedge^r \phi(e_{i_1} \wedge \ldots \wedge e_{i_r}) = \sum_{1 \le j_1 < \ldots < j_r \le m} M(j_1, \ldots, j_r \,|\, i_1, \ldots, i_r; \phi) f_{j_1} \wedge \ldots \wedge f_{j_r},$$

where $M(j_1, \ldots, j_r \,|\, i_1, \ldots, i_r; \phi)$ denotes the $r \times r$ minor of the matrix $(\phi_{j,i})$ corresponding to the rows $j_1, \ldots, j_r$ and columns $i_1, \ldots, i_r$.

The vector space

$$\bigwedge^{\bullet}(E) := \bigoplus_{r \ge 0} \bigwedge^r E$$

has a natural multiplication

$$m : \bigwedge^{\bullet}(E) \otimes \bigwedge^{\bullet}(E) \to \bigwedge^{\bullet}(E)$$

given by the formula

$$m(u_1 \wedge \ldots u_r \otimes v_1 \wedge \ldots \wedge v_s) = u_1 \wedge \ldots \wedge u_r \wedge v_1 \wedge \ldots \wedge v_s.$$

This gives $\bigwedge^\bullet(E)$ the structure of associative, graded commutative algebra (meaning that the commutative law reads $fg = (-1)^{\deg(f)\deg(g)} gf$). We call this algebra *the exterior algebra on* E. The algebra $\bigwedge^\bullet(E)$ has a unit $\eta : \mathbf{K} \to \bigwedge^\bullet(E)$.

The components of the multiplication map will be denoted by $m : \bigwedge^r E \otimes \bigwedge^s E \to \bigwedge^{r+s} E$.

The diagonal map $\Delta : E \to E \oplus E$ induces an algebra map

$$\Delta : \bigwedge^{\bullet}(E) \to \bigwedge^{\bullet}(E \oplus E) \cong \bigwedge^{\bullet}(E) \otimes \bigwedge^{\bullet}(E)$$

which we will call *the diagonal* (or *comultiplication*) *map*.

The components of Δ will be denoted by $\Delta : \bigwedge^{r+s} E \to \bigwedge^r E \otimes \bigwedge^s E$. In terms of elements we have

$$\begin{aligned} &\Delta\,(u_1 \wedge \ldots \wedge u_{r+s}) \\ &\quad = \sum_{\sigma \in \Sigma_{r+s}^{r,s}} (-1)^{\operatorname{sgn}\sigma} u_{\sigma(1)} \wedge \ldots \wedge u_{\sigma(r)} \otimes u_{\sigma(r+1)} \wedge \ldots \wedge u_{\sigma(r+s)} \end{aligned}$$

where $\Sigma_{r+s}^{r,s} = \{\sigma \in \Sigma_{r+s} \mid \sigma(1) < \ldots < \sigma(r);\;\; \sigma(r+1) < \ldots < \sigma(r+s)\}$.

Finally we have the counit map

$$\epsilon : \bigwedge^{\bullet}(E) \to \mathbf{K},$$

defined to be zero on all spaces $\bigwedge^r E$ for $r > 0$, and satisfying $\epsilon\eta(1) = 1$.

The following proposition is an elementary calculation.

(1.1.2) Proposition.

(a) The maps $m, \Delta, \epsilon, \eta$ define on $\bigwedge^\bullet(E)$ the structure of commutative, cocommutative bialgebra.

(b) The map $\alpha : \bigwedge^\bullet(E^*) \to (\bigwedge^\bullet E)^*$ defined in (1.1.1) (c) is an isomorphism of bialgebras.

Part (b) of the proposition means that the dual map to the multiplication map m on $\bigwedge^\bullet(E)$ is the diagonal map Δ on $\bigwedge^\bullet(E^*)$ and vice versa.

We define *the r-th symmetric power $S_r E$ of E* to be the r-th tensor power $E^{\otimes r}$ of E divided by the submodule generated by the elements

$$u_1 \otimes \ldots \otimes u_r - u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(r)}$$

for all $\sigma \in \Sigma_r$, $u_1, \ldots, u_r \in E$. We denote the coset of $u_1 \otimes \ldots \otimes u_r$ by $u_1 \ldots u_r$.

The following basic properties of symmetric powers are proved in [L, chapter XVI, section 8].

(1.1.3) Proposition.

(a) Let $\{e_1, \ldots, e_n\}$ be an ordered basis of E. Then the elements $e_1^{i_1} \ldots e_n^{i_n}$ for $i_1 + \ldots + i_n = r$ form a basis of $S_r E$. In particular $S_r E$ is a free **K***-module of dimension $\binom{n+r-1}{r}$.*

(b) (Universality property of symmetric powers) We have a functorial isomorphism

$$\theta_M : \mathrm{Sym}^r(E^r, M) \to \mathrm{Hom}_{\mathbf{K}}(S_r E, M)$$

where $\mathrm{Sym}^r(E^r, M)$ denotes the set of multilinear symmetric maps from $E^{\times r}$ to M, given by the formula $\theta_M^r(f)(u_1 \ldots u_r) = f(u_1, \ldots, u_r)$.

The r-th symmetric power is an endofunctor on the category of free **K**-modules and linear maps. More precisely, for two free **K**-modules E, F and a linear map $\phi : E \to F$ we have a well-defined linear map

$$S_r\phi : S_r E \to S_r F$$

defined by the formula $S_r\phi(u_1 \ldots u_r) = \phi(u_1) \ldots \phi(u_r)$. Let us denote $m = \dim F$. Let $\{e_1, \ldots, e_n\}$ be a basis of E, and let $\{f_1, \ldots, f_m\}$ be a basis of F. In these bases ϕ correspond to the $m \times n$ matrix $(\phi_{j,i})$ where

$$\phi(e_i) = \sum_{j=1}^{m} \phi_{j,i} f_j.$$

The map $S_r\phi$ can be written in the corresponding bases of $S_r E$, $S_r F$ as follows:

$$S_r\phi(e_{i_1} \ldots e_{i_r}) = \sum_{1 \le j_1 < \ldots < j_r \le m} P(j_1, \ldots, j_r | \, i_1, \ldots, i_r; \phi) f_{j_1} \ldots f_{j_r},$$

where $P(j_1, \ldots, j_r | \, i_1, \ldots, i_r; \phi)$ denotes the permanent of the $r \times r$ submatrix of the matrix $(\phi_{j,i})$ corresponding to the (possibly repeated) rows $j_1, \ldots, j_r$ and (possibly repeated) columns $i_1, \ldots, i_r$. More precisely, if the columns $(i_1, \ldots, i_r)$ with repetitions are written as $i_1^{b_1}, \ldots, i_s^{b_s}$ with $b_1 + \ldots + b_s = r$, we have

$$P(j_1, \ldots, j_r | \, i_1^{b_1}, \ldots, i_s^{b_s}) = \sum_{\sigma \in \Sigma_r/(\Sigma_{b_1} \times \Sigma_{b_s})} \phi(j_1, i_{\sigma(1)}) \ldots \phi(j_r, i_{\sigma(r)}).$$

where $\Sigma_{b_1} \times \ldots \times \Sigma_{b_s}$ is the subgroup of permutations from Σ_r preserving the groups of repeating symbols among $j_1, \ldots, j_r$.

The vector space

$$\mathrm{Sym}(E) := \bigoplus_{r \ge 0} S_r E$$

has a natural multiplication

$$m : \mathrm{Sym}(E) \otimes \mathrm{Sym}(E) \to \mathrm{Sym}(E)$$

given by the formula

$$m(u_1 \dots u_r \otimes v_1 \dots v_s) = u_1 \dots u_r v_1 \dots v_s.$$

This gives $\mathrm{Sym}(E)$ the structure of associative, commutative algebra. We call this algebra *the symmetric algebra on* E. It can be identified with the polynomial ring over $\mathbf{K}$ in n variables $e_1, \dots, e_n$. In order to keep the notion of commutativity the same as for the exterior algebras, we assume that $\mathrm{Sym}(E)$ is generated by elements of degree 2.

The components of the multiplication map will be denoted by $m : S_r E \otimes S_s E \to S_{r+s} E$.

We also have an obvious unit map $\eta : \mathbf{K} \to \mathrm{Sym}(E)$ sending $\mathbf{K}$ to the degree zero component of $\mathrm{Sym}(E)$.

The diagonal map $\Delta : E \to E \oplus E$ induces an algebra map

$$\Delta : \mathrm{Sym}(E) \to \mathrm{Sym}(E \oplus E) \cong \mathrm{Sym}(E) \otimes \mathrm{Sym}(E),$$

which we will call *the diagonal (or comultiplication) map*.

The components of Δ will be denoted by $\Delta : S_{r+s} E \to S_r E \otimes S_s E$. In terms of elements we have

$$\Delta(u_1 \dots u_{r+s}) = \sum_{\sigma \in \Sigma^{r,s}_{r+s}} u_{\sigma(1)} \dots u_{\sigma(r)} \otimes u_{\sigma(r+1)} \dots u_{\sigma(r+s)}$$

where $\Sigma^{r,s}_{r+s} = \{\sigma \in \Sigma_{r+s} \,|\sigma(1) < \dots < \sigma(r); \sigma(r+1) < \dots < \sigma(r+s)\}$.

Finally we have the counit map

$$\epsilon : \mathrm{Sym}(E) \to \mathbf{K}$$

defined to be zero on all spaces $S_r E$ for $r > 0$, and satisfying $\epsilon\eta(1) = 1$.

We have the following analogue of (1.1.2) (a).

(1.1.4) Proposition. *The maps* $m, \Delta, \epsilon, \eta$ *define on* $\mathrm{Sym}(E)$ *the structure of a commutative, cocommutative bialgebra.*

Let us investigate the duality. The algebra $\mathrm{Sym}(E) = \bigoplus_{r \geq 0} S_r E$ is not finite dimensional, so instead of the dual we have to work with the graded dual

$$\mathrm{Sym}(E)^*_{gr} := \bigoplus_{r \geq 0} (S_r E)^*.$$

The module map

$$E^* = (S_1 E)^* \to \mathrm{Sym}(E)^*_{gr}$$

induces by universality an algebra map

$$\beta : \mathrm{Sym}(E^*) \to \mathrm{Sym}(E)^*_{gr}.$$

This map β is an isomorphism only when $\mathbf{K}$ contains a field of rational numbers. In fact it is given by the formula

$$\beta(l_1 \dots l_r)(u_1 \dots u_r) = \sum_{\sigma \in \Sigma_r} l_{\sigma(1)}(u_1) \dots l_{\sigma(r)}(u_r).$$

In particular, when $l_1 = \dots = l_r,\ u_1 = \dots = u_r$ we see that $\beta(l_1^r) = r!(u_1^r)^*$.

In order to describe the graded dual of the symmetric algebra we introduce the divided powers.

We define *the r-th divided power $D_r(E)$* as the dual of the symmetric power.

$$D_r(E) := (S_r(E^*))^*.$$

Its basis is the dual basis to the natural basis of the symmetric power. If $\{e_1, \dots, e_n\}$ is a basis of E, we define $e_1^{(i_1)} \dots e_n^{(i_n)}$ to be the element of the dual basis to the basis $\{(e_1^*)^{j_1} \dots (e_n^*)^{j_n}\}$, dual to $(e_1^*)^{i_1} \dots (e_n^*)^{i_n}$.

For every $u \in E$ we can define its *r-th divided power $u^{(r)} \in D_r E$*. It is given by the formula

$$\left(\sum_{i=1}^n u_i e_i\right)^{(r)} = \sum_{p_1+\dots+p_n=r} u_1^{p_1} \dots u_n^{p_n} e_1^{(p_1)} \dots e_n^{(p_n)}.$$

It is easy to check that this definition does not depend on the choice of basis $\{e_1, \dots, e_n\}$.

(1.1.5) Proposition. *The divided powers have the following properties:*

(a) $u^{(0)} = 1,\ u^{(1)} = u,\ u^{(r)} \in D_r E,$
(b) $u^{(p)} u^{(q)} = \binom{p+q}{q} u^{(p+q)},$
(c) $(u+v)^{(p)} = \sum_{k=0}^{p} u^{(k)} v^{(p-k)},$
(d) $(uv)^{(p)} = u^{(p)} v^{(p)},$
(e) $(u^{(p)})^{(q)} = [p,q] u^{(pq)}$ *for* $u \in E$; $[p,q] = [(pq)!]/(q!p^q!)$.

(1.1.6) Remark. *In the notation used above, $e_1^{(i_1)} \dots e_n^{(i_n)}$ has a double meaning. It is the element of the dual basis to the basis in the symmetric power,* *and it is the product of divided powers. It is not difficult to see that the two elements coincide.*

The r-th divided power is an endofunctor on the category of free $\mathbf{K}$-modules and linear maps. More precisely, for two free $\mathbf{K}$-modules E, F and a linear map $\phi : E \to F$ we have a well-defined linear map

$$D_r\phi : D_r E \to D_r F$$

which is best described as the transpose of the map $S_r(\phi^*) : S_r(F^*) \to S_r(E^*)$. This also gives the description of the matrix coefficients for $D_r\phi$ as polynomials in the entries of ϕ, which we leave to the reader.

The divided power algebra $D(E) := \bigoplus D_r(E)$ on E is a commutative, cocommutative algebra because it is a graded dual of the symmetric algebra on E^*. Again we denote the components of the multiplication map by

$$m : D_r E \otimes D_s E \to D_{r+s} E,$$

and the components of the comultiplication by

$$\Delta : D_{r+s} E \to D_r E \otimes D_s E.$$

Let us record the duality statements.

(1.1.7) Proposition.

(a) The multiplication map

$$m : D_r E \otimes D_s E \to D_{r+s} E$$

is the dual of the diagonal map

$$\Delta : S_{r+s} E^* \to S_r E^* \otimes S_s E^*.$$

(b) The diagonal map

$$\Delta : D_{r+s} E \to D_r E \otimes D_s E$$

is the dual of the multiplication map

$$m : S_r E^* \otimes S_s E^* \to S_{r+s} E^*.$$

(c) The diagonal map $\Delta : D_{r+s} E \to D_r E \otimes D_s E$ is given by the formula

$$\Delta(e_1^{(i_1)} \dots e_n^{(i_n)}) = \sum_{j_1+\dots+j_n=r,\ \ 0\le j_s\le i_s \text{ for } s=1,\dots,n} e_1^{(j_1)} \dots e_n^{(j_n)} \otimes e_1^{(i_1-j_1)} \dots e_n^{(i_n-j_n)}.$$

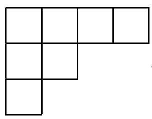
1.1.2. Partitions, Skew Partitions. Combinatorics of $\mathbf{Z}_2$-Graded Tableaux.

Let n be a natural number. *A partition* λ of n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_s)$ of natural numbers such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_s = n$. We identify the partitions $(\lambda_1, \ldots, \lambda_s)$ and $(\lambda_1, \ldots, \lambda_s, 0)$. To each partition λ we associate its Young frame (or Ferrers diagram) $D(\lambda)$. It can be defined as

$$D(\lambda) = \{(i, j) \in \mathbf{Z} \times \mathbf{Z} \mid (1 \leq i \leq s, 1 \leq j \leq \lambda_i \}.$$

To represent the Young frames graphically we think of them as contained in the fourth quadrant. A Young frame is a set of boxes with λ_i boxes in the i-th row from the top. Formally it could be achieved by considering the point $(j, -i)$ instead of (i, j).

(1.1.8) Example. $\lambda = (4, 2, 1)$:

$D(\lambda) =$  .

Formally the boxes of $D((4, 2, 1))$ *correspond to the set of points*

$$\{(1, -1), (2, -1), (3, -1), (4, -1), (1, -2), (2, -2), (1, -3)\}$$

in the grid $\mathbf{Z} \times \mathbf{Z}$.

Let λ be a partition. We say that λ has *a Durfee square of size* r (or rank $\lambda = r$) if $\lambda_r \geq r$, $\lambda_{r+1} \leq r$, i.e., if the biggest square fitting inside of λ is an $r \times r$ square.

Let λ be a partition, and let X be a box in λ. The set of boxes to the right of X (including X) is called an *arm of* X. The set of boxes below X (including X) is called the *leg of* X. The arm length (leg length) of X are defined as the numbers of boxes in the arm (leg) of X.

The arm and leg of X form *a hook* of X. The number of boxes in the hook of X is called the hook length of X.

Let λ be a partition of rank r. Let a_i (b_i) be the arm length (leg length) of the i-th box on the diagonal of λ. The partition λ is uniquely determined by its rank r and the numbers a_i, b_i $(1 \leq i \leq r)$. These numbers satisfy the conditions $a_1 > \ldots > a_r > 0$, $b_1 > \ldots, b_r > 0$.

We will sometimes denote by $\lambda = (a_1, \ldots, a_r | b_1, \ldots, b_r)$ the partition with diagonal arm lengths a_i and diagonal leg lengths b_i. We refer to this as a *Frobenius* (or *hook*) *notation* for λ.

(1.1.9) Example. *The partition* $\lambda = (4, 3, 2)$ *in the hook notation is* $(4, 2|3, 2)$. *The boxes in the arm (leg) of the i-th diagonal box are filled with symbol i* ($\bar{i}$)*:*

X	1	1	1
$\bar{1}$	X	2	
$\bar{1}$	$\bar{2}$		

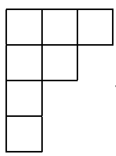
.

Let λ be a partition. The conjugate (or dual) partition λ' is defined by setting

$$\lambda'_i = \mathrm{card}\{t \,|\lambda_t \geq i\}.$$

The Young frame of λ' is obtained from the Young frame of λ by reflecting in the line $y = -x$.

(1.1.10) Example. $\lambda = (4, 2, 1)$, $\lambda' = (3, 2, 1, 1)$:

$D(\lambda') =$  .

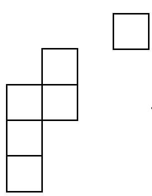
Let λ and μ be two partitions. We say that μ is contained in λ (denoted $\mu \subset \lambda$) if for each i we have $\mu_i \leq \lambda_i$. Let λ and μ be two partitions with $\mu \subset \lambda$. We refer to such a pair as a *skew partition* λ/μ.

We associate to a skew partition λ/μ the skew Young frame

$$D(\lambda/\mu) := D(\lambda) \setminus D(\mu).$$

Graphically we can represent it as a Young frame of λ with the boxes corresponding to μ missing.

(1.1.11) Example. $\lambda = (4, 2, 2, 1, 1)$, $\mu = (3, 1)$:

$D(\lambda/\mu) =$  .

Let $A = (A_0, A_1)$ be a $\mathbf{Z}_2$-graded set, i.e. the pair of sets indexed by $\{0, 1\}$. Assume that the set A is ordered by a total order $\lhd$. A *tableau of shape* λ/μ *with values in* A is a function $T : D(\lambda/\mu) \to A$.

(1.1.12) Definition.

(a) A tableau T of shape λ/μ with values in A is row standard if for each (u, v) we have $T(u, v) \lhd T(u, v+1)$ with equality possible if $T(u, v) \in A_1$.

(b) We say that a tableau T of shape λ/μ with values in A is column standard if $T(u, v) \lhd T(u+1, v)$ with equality possible when $T(u, v) \in A_0$.

(c) A tableau T of shape λ/μ with values in A is standard if it is both column standard and row standard.

(1.1.13) Notation. *We denote by* $\mathrm{RST}(\lambda/\mu, A)$ *(*$\mathrm{CST}(\lambda/\mu, A)$, $\mathrm{ST}(\lambda/\mu, A)$*) the set of row standard (column standard, standard) tableaux of shape* λ/μ *with values in* A*. We denote by* $[1, m] \cup [1, n]'$ *the* $\mathbf{Z}_2$*-graded set* A *with* $A_0 = [1, m]$, $A_1 = [1', n']$ *and with the order* $\lhd$ *defined to be the natural order on* A_0 *and* A_1 *with* A_0 *preceeding* A_1*. Similarly we define the* $\mathbf{Z}_2$*-graded set* $[1, n]' \cup [1, m]$ *(here* A_1 *preceeds* A_0*).*

(1.1.14) Examples. *Let* $\lambda = (4, 2, 2, 1, 1)$, $\mu = (2, 1)$*. Let* $A = [1, 2] \cup [1, 3]'$.

(a) The tableau

$$T_1 = \begin{array}{cccc} & & 1 & 2 \\ & 1' & & \\ 1' & 1' & & \\ 2' & & & \\ 2' & & & \end{array}$$

is row standard but not column standard.

(b) The tableau

$$T_2 = \begin{array}{cccc} & & 1 & 1 \\ & 1' & & \\ 1' & 2' & & \\ 2' & & & \\ 3' & & & \end{array}$$

is column standard but not row standard.

(c) The tableau

$$T_3 = \begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline & 1' & & \\ \hline 1' & 2' & & \\ \hline 2' & & & \\ \hline 3' & & & \\ \hline \end{array}$$

is standard.

Let λ/μ be a skew partition, and let $A = (A_0, A_1)$ be a $\mathbf{Z}_2$-graded set ordered by the total order $\lhd$. We define the orders $\preceq$ (relative to $\lhd$) on the sets of row standard (column standard, standard) tableaux as follows.

Consider the set $\mathrm{RST}(\lambda/\mu, A)$. Given two tableaux T, U, we have $T \preceq U$ if $T = U$. Assume that $T \neq U$. Let us write them as $T = (T_1, \dots, T_s)$, $U = (U_1, \dots, U_s)$ with T_i (U_i) being the part of T (U) from the i-th row of λ/μ. Let j be the minimal i for which $T_i \neq U_i$. We have $T_j = (T(j, 1), \dots, T(j, \lambda_j - \mu_j))$, $U_j = (U(j, 1), \dots, U(j, \lambda_j - \mu_j))$. Now let k be the smallest index for which $T(j, k) \neq U(j, k)$ (such a k exists by the choice of j). We say that $T \preceq U$ if and only if $T(j, k) \lhd U(j, k)$.

The order $\preceq$ on $\mathrm{ST}(\lambda/\mu, A)$ is defined to be the restriction of $\preceq$ from $\mathrm{RST}(\lambda/\mu, A)$.

Finally we define the order $\preceq$ on $\mathrm{CST}(\lambda/\mu, A)$. Given two tableaux T, U from $\mathrm{CST}(\lambda/\mu, A)$, then $T = U$ implies $T \preceq U$. Assume $T \neq U$. We write $T = (T^1, \dots, T^s)$, $U = (U^1, \dots, U^s)$ with T^i (U^i) being the part of T (U) from the i-th column of λ/μ. Let j be the minimal i for which $T^i \neq U^i$. We have $T^j = (T(1, j), \dots, T(\lambda'_j - \mu'_j, j))$, $U^j = (U(1, j), \dots, U(\lambda'_j - \mu'_j, j))$. Now let k be the smallest index for which $T(k, j) \neq U(k, j)$ (such k exists by the choice of j). We say that $T \preceq U$ if and only if $T(k, j) \lhd U(k, j)$.

Note. *The order* $\preceq$ *on* $\mathrm{ST}(\lambda/\mu, A)$ *is the restriction of the order* $\preceq$ *on* $\mathrm{RST}(\lambda/\mu, A)$. *It is different from the restriction of* $\preceq$ *on* $\mathrm{CST}(\lambda/\mu, A)$.

(1.1.15) Examples. *Let* $\lambda = (2, 1)$, $\mu = (0)$. *Set* $A = [1, 2] \cup [1, 2]'$. *In* $\mathrm{RST}(\lambda/\mu, A)$ *we have*

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 1' & \\ \hline \end{array} \preceq \begin{array}{|c|c|} \hline 1 & 1' \\ \hline 2 & \\ \hline \end{array} \preceq \begin{array}{|c|c|} \hline 2 & 1' \\ \hline 1 & \\ \hline \end{array}.$$

In CST$(\lambda/\mu, A)$ *we have*

$$\begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2 \\ \cline{1-1}\end{array} \preceq \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1' \\ \cline{1-1}\end{array} \preceq \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1' \\ \cline{1-1}\end{array}.$$

In ST$(\lambda/\mu, A)$ *we have*

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1' \\ \cline{1-1}\end{array} \preceq \begin{array}{|c|c|}\hline 1 & 1' \\ \hline 2 \\ \cline{1-1}\end{array}.$$

1.2. Homological and Commutative Algebra

1.2.1. Regular Sequences, Koszul Complexes, Depth

Let R be a commutative Noetherian ring. Let M be an R-module. *The dimension* $\dim M$ of M is defined to be the Krull dimension of $R/\mathrm{Ann}(M)$, where

$$\mathrm{Ann}(M) = \{x \in R \mid xM = 0\}$$

is the annihilator of M.

Let I be an ideal in R. If $IM \neq M$, we define the I-depth of M as

$$\mathrm{depth}_R(I, M) = \min\{i \mid \mathrm{Ext}^i_R(I, M) \neq 0\}.$$

In the case $IM = M$ we define $\mathrm{depth}_R(I, M) = \infty$. For a finitely generated R-module M we have $IM \neq M$ if and only if $\mathrm{depth}_R(I, M) < \infty$ if and only if $\mathrm{depth}_R(I, M) \leq \dim M$.

A sequence $\underline{a} = (a_1, \ldots, a_n)$ of elements from R is an M-sequence (or a regular sequence on M) if $M \neq (a_1, \ldots, a_n)M$ and the multiplication $a_i : M_{i-1} \to M_{i-1}$ is injective for $i = 0, 1, \ldots, n-1$, where $M_i := M/(a_1, \ldots, a_i)M$.

The connection between these notions is expressed in

(1.2.1) Theorem. *Let R be a Noetherian ring, and M a finitely generated R-module. Let I be an ideal in R. The following conditions are equivalent:*

(a) $\mathrm{depth}_R(I, M) \geq n$.
(b) $\mathrm{Ext}^i_R(R/I, M) = 0$ *for* $i < n$.
(c) *There exists an M-sequence $\underline{a} = (a_1, \ldots, a_n)$ of length n with $a_i \in I$ for $i = 1, \ldots, n$.*

A regular sequence $(a_1, \ldots, a_n)$ is *a maximal regular M-sequence* if there is no b such that $(a_1, \ldots, a_n, b)$ is an M-sequence. In particular the theorem implies that two maximal regular M-sequences with terms from I must have the same length, equal to depth(I, M).

Let M be an R-module, and let $\underline{a} = (a_1, \ldots, a_n)$ be a sequence of elements from R. We define the *Koszul complex* $K(\underline{a}, M)_\bullet$ as follows. For an n-dimensional free R-module $E = R^n$ with a basis $e_1, \ldots, e_n$ we set $K(\underline{a}, M)_i = \bigwedge^i E \otimes_R M$, and the differential

$$d : \bigwedge^{i} E \otimes_R M \to \bigwedge^{i-1} E \otimes_R M$$

is defined by the formula

$$d(e_{j_1} \wedge \ldots \wedge e_{j_i} \otimes m) = \sum_{u=1}^{i} (-1)^{u+1} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_u} \wedge \ldots \wedge e_{j_i} \otimes a_{j_u} m.$$

Let M be a finitely generated R-module. We define the codimension of M,

$$\operatorname{codim}_R(M) := \operatorname{ht} \operatorname{Ann}(M),$$

where ht denotes the height of an ideal. We also define the grade of M,

$$\operatorname{grade}_R(M) = \operatorname{depth}_R(\operatorname{Ann}(M), R).$$

The homological properties of Koszul complex include the information about the depth.

(1.2.2) Theorem. *Let M be a finitely generated R-module, and let $\underline{a} = (a_1, \ldots, a_n)$ be a sequence of elements from R. Denote $I = (a_1, \ldots, a_n)$. Then*

$$\operatorname{depth}_R(I, M) = n - \max\{\, i \mid H_i(K(\underline{a}, M)) \neq 0 \,\}.$$

(1.2.3) Corollary. *Let R be a commutative ring. Assume that $I = (a_1, \ldots, a_n)$ is an ideal generated by a regular sequence. Then the Koszul complex $K(\underline{a}, R)_\bullet$ is a free resolution of the R-module R/I.*

The ideal I is a *complete intersection ideal of codimension n* if there exists a regular sequence $(a_1, \ldots, a_n)$ such that $I = (a_1, \ldots, a_n)$. Thus the finite free resolutions of complete intersection ideals are provided by Koszul complexes.

The projective dimension, codimension, and grade of an R-module are related.

(1.2.4) Theorem. *For an R-module $M \neq 0$ we have*

$$\mathrm{pd}_R(M) \geq \mathrm{codim}(M) \geq \mathrm{grade}_R(M).$$

A finitely generated R-module N is *perfect* if $\mathrm{pd}_R(N) = \mathrm{grade}(N)$. In that case the inequalities in (1.2.4) become equalities. We call $\mathrm{codim}(N)$ the *codimension of N*. Sometimes by abuse of notation we call the grade of R/I the *grade of the ideal I*.

Let us note the following consequence of Theorem (1.2.4) applied to $N = R$.

(1.2.5) Proposition. *Let I be an ideal of codimension n. The functor $N \mapsto \mathrm{Ext}^n_R(N, R)$ is an exact contravariant involution on the category of perfect modules N with $Ann(N) = I$ up to radical.*

If R/I is a perfect module, we call I a *perfect ideal*. An ideal I is *Gorenstein* if I is perfect and $\mathrm{Ext}^n(R/I, R) \cong R/I$ for $n = \mathrm{codim}(R/I)$.

1.2.2. Cohen–Macaulay Rings and Modules, Gorenstein Rings

Let (R, m) be a local ring. The depth of a module M is defined as

$$\mathrm{depth}_R(M) := \mathrm{depth}_R(m, M).$$

We have the following inequalities:

(1.2.6) Proposition. *Let M be a finitely generated module over a local ring R. Then*

$$\mathrm{depth}(M) \leq \dim M \leq \dim R.$$

An R-module M is *Cohen–Macaulay* if $\mathrm{depth}(M) = \dim M$. If depth $(M) = \dim R$, we say that M is *maximal Cohen–Macaulay*. The zero module is by definition maximal Cohen–Macaulay. The ring R is Cohen–Macaulay if it is Cohen–Macaulay as a module over itself.

The projective dimension and depth of a module over a local ring are complementary to each other.

(1.2.7) Theorem (Auslander–Buchsbaum Formula). *Let R be a Noetherian local ring. Assume that $\mathrm{pd}_R(M) < \infty$. Then we have*

$$\mathrm{pd}_R(M) + \mathrm{depth}(M) = \mathrm{depth}(R).$$

It follows that if R is Cohen–Macaulay and M is a maximal Cohen–Maculay module of finite projective dimension over R, then M is R-free.

If R is a Cohen–Macaulay local ring and M is a finitely generated R-module of finite projective dimension, then M is Cohen–Macaulay if and only if it is perfect.

If R is a Cohen–Macaulay local ring, then $\dim R = \dim R/P$ for every associated prime P of R. This means that R is equidimensional.

A local ring (R, m) is *Gorenstein* if an only if R has a finite injective dimension as an R-module.

(1.2.8) Theorem. *Let (R, m) be a local ring of dimension d. The following conditions are equivalent:*

(a) R is Gorenstein,
(b) for $i \neq d$ we have $\operatorname{Ext}^i_R(K, R) = 0$, $\operatorname{Ext}^d(K, R) = K$,
(c) there exists $i > d$ such that $\operatorname{Ext}^i_R(K, R) = 0$,
(d) $\operatorname{Ext}^i_R(K, R) = 0$ *for $i < d$,* $\operatorname{Ext}^d_R(K, R) = K$,
(e) R is Cohen–Macaulay and $\operatorname{Ext}^d_R(K, R) = K$.

Recall that the *embedding dimension* of a local ring is $\operatorname{emdim}(R) = \dim_K m/m^2$. A local ring R is *regular* if $\operatorname{emdim}(R) = \dim R$.

We denote by gl dim R *the global dimension* of R.

(1.2.9) Theorem (Auslander and Buchsbaum, Serre). *Let (R, m) be a local ring of dimension d. Then the following are equivalent:*

(a) R is a regular local ring,
(b) $\operatorname{gl.dim} R < \infty$,
(c) $\operatorname{gl.dim} R = d$,
(d) $\operatorname{pd}_R K = d$,
(e) m is generated by a regular sequence of length d.

The connection between the notions of Cohen–Macaulay (Gorenstein) ring and perfect (Gorenstein) ideal is stated in the next proposition.

(1.2.10) Proposition.
(a) Let R be a Cohen–Macaulay local ring. Then the ring R/I is Cohen–Macaulay if and only if I is perfect,
(b) Let R be a Gorenstein local ring. Then R/I is Gorenstein if and only if I is a Gorenstein ideal.

The theory outlined above for local rings has an analogue for graded rings and graded modules. Let us state the corresponding statements.

Let R be a graded ring $R = \bigoplus_{i\geq 0} R_i$ where $R_0 = K$ is a field and R_i are finite dimensional vector spaces over K. We assume that R is generated as a K-algebra by elements of degree 1, which implies that R is Noetherian. We denote by m the maximal ideal $m = R_+ = \bigoplus_{i>0} R_i$. For a graded R-module M we denote $\operatorname{depth}_R(M) := \operatorname{depth}_R(m, M)$.

Then the following statements hold.

(1.2.6)′ Proposition. *Let M be a finitely generated graded module over a graded ring R. Then*

$$\operatorname{depth}_R(M) \leq \dim M \leq \dim R.$$

(1.2.7)′ Theorem (Auslander–Buchsbaum formula). *Let R be a graded ring, and let M be a graded R-module. Assume that* $\operatorname{pd}_R(M) < \infty$. *Then we have*

$$\operatorname{pd}_R(M) + \operatorname{depth}_R(M) = \operatorname{depth}_R(R).$$

(1.2.8)′ Theorem. *Let R be a graded ring of dimension d with the maximal ideal $m = R^+$. The following conditions are equivalent:*

(a) R is Gorenstein,
(b) for $i \neq d$ we have $\operatorname{Ext}^i_R(K, R) = 0$, $\operatorname{Ext}^d(K, R) = K$,
(c) there exists $i > d$ such that $\operatorname{Ext}^i_R(K, R) = 0$,
(d) $\operatorname{Ext}^i_R(K, R) = 0$ *for $i < d$,* $\operatorname{Ext}^d_R(K, R) = K$,
(e) R is Cohen–Macaulay and $\operatorname{Ext}^d_R(K, R) = K$.

The theorem characterizing the regular rings differs because the only graded regular ring is a polynomial ring.

The embedding dimension of a graded ring R is $\operatorname{emdim}(R) = \dim_K m/m^2$. A graded ring R is *regular* if $\operatorname{emdim}(R) = \dim R$.

(1.2.9)′ Theorem. *Let R be a graded ring of dimension d with the maximal ideal $m = R^+$. Then the following are equivalent:*

(a) R is a regular graded ring,
(b) $\operatorname{gl.dim} R < \infty$,
(c) $\operatorname{gl.dim} R = d$,
(d) $\operatorname{pd}_R K = d$,
(e) m is generated by a regular sequence of length d,
(f) R is a polynomial ring over K in d variables.