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IN MATHEMATICS | 84

An Introduction to Quiver Representations

Harm Derksen
Jerzy Weyman



American Mathematical Society

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IN MATHEMATICS **184**

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American Mathematical Society
Providence, Rhode Island

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2010 *Mathematics Subject Classification*. Primary 16G20, 16G10, 16G70, 14L24, 13A50.

For additional information and updates on this book, visit
www.ams.org/bookpages/gsm-184

Library of Congress Cataloging-in-Publication Data

Derksen, Harm, 1970- | Weyman, Jerzy, 1955-

Title: An introduction to quiver representations / Harm Derksen, Jerzy Weyman.

Description: Providence, Rhode Island : American Mathematical Society, [2017] | Series: Graduate studies in mathematics ; volume 184 | Includes bibliographical references and index.

Identifiers: LCCN 2017016426 | ISBN 9781470425562 (alk. paper)

Subjects: LCSH: Directed graphs. | Representations of graphs. | Vector spaces. | AMS: Associative rings and algebras – Representation theory of rings and algebras – Representations of Artinian rings. msc | Associative rings and algebras – Representation theory of rings and algebras – Auslander-Reiten sequences (almost split sequences) and Auslander-Reiten quivers. msc | Algebraic geometry – Algebraic groups – Geometric invariant theory. msc | Commutative algebra – General commutative ring theory – Actions of groups on commutative rings; invariant theory. msc

Classification: LCC QA166.15 .D47 2017 | DDC 512/.46–dc23

LC record available at <https://lcn.loc.gov/2017016426>

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Preface

The idea of writing this book came from the collaboration of the authors. We started working on quiver representation when the first author visited Northeastern University as a Research Scholar in the academic year 1997/1998. We both came from other fields of algebra so we had to learn the subject from scratch. At the same time we were interested in connections of quiver representations with invariant theory, representations of algebraic groups, and algebraic geometry.

This experience made us realize that (except for Crawley-Boevey's notes [18] available online), there is no introductory text which would allow a person without any knowledge of Artin algebras to learn the subject quickly.

Later both authors taught courses on quiver representations at University of Michigan and Northeastern University, respectively. The second author also taught a two month course on quiver representations at Tor Vergata University in Rome in the spring of 2007.

We are grateful to Calin Chindris, Jiarui Fei, Ryan Kinser, Andrea Appel, Andrew Carroll, Sachin Gautam, Daniel Labardini, Kavita Sutar, Salvatore Stella, Riccardo Aragona, Ada Boralevi, Cristina DiTrapano, Luca Moci, and other students who took part in these courses.

The book is addressed to non-specialists, who want to learn the subject without going through the extended preparation in algebra, just starting from basic linear algebra. It turned out to be impossible to be completely elementary, so in some places we use some basic algebraic geometry (mainly the dimension counts).

We do not prove the results in full generality, working over the field \mathbb{C} of complex numbers. We also work mostly with acyclic quivers. We only cover Auslander-Reiten duality in the case of hereditary algebras.

The book reflects our point of view, so the semi-invariants are covered in detail and we stress their role in our approach. Some of the results could be proved just using stability conditions without mentioning semi-invariants, but we find the combinatorics of the rings of semi-invariants quite fascinating.

Still many important topics are left out, for example, Ringel-Hall algebras and Nakajima quiver varieties.

In recent years the field developed very quickly. New concepts and connections emerged. We wanted this development to be reflected in the book, hence there are chapters on orthogonal categories, exceptional sequences and cluster categories.

We stress the connections of quiver representations with representations of algebraic groups and moduli problems.

The authors thank the National Science Foundations for its support during the writing of this book.

We discussed the subject of the book with many mathematicians, with some of whom we coauthored our papers. We benefited from discussions with Prakash Belkale, Calin Chindris, Bill Crawley-Boevey, Jose Antonio de la Pena, Jiarui Fei, Lutz Hille, Kiyoshi Igusa, Ryan Kinser, Mark Kleiner, Visu Makam, Kent Orr, Charles Paquette, Idun Reiten, Claus Ringel, Ralf Schiffler, Aidan Schofield, Gordana Todorov, Michel Van den Bergh, and Andrei Zelevinsky.

The second author also wants to thank Piotr Dowbor, Daniel Simson and Andrzej Skowroński whose questions about the invariant theory of quivers aroused his interest in the subject.

Notation

- $[1, n] = \{1, 2, \dots, n\}$, 195
 $\langle \alpha, \beta \rangle$, Euler form, 32
 (α, β) , Cartan form, 50
 $\text{add}(T)$, additive subcategory, 254
 $|\alpha|$, 3
 $\alpha \rightarrow \beta$, 209
 $\alpha^{\oplus m}$, 250
 $A\text{-mod}$, 12
 $\mathbf{A}_n, \mathbf{D}_n, \mathbf{E}_n$, Dynkin graphs, 52
 $\widehat{\mathbf{A}}_n, \widehat{\mathbf{D}}_n, \widehat{\mathbf{E}}_n$, extended Dynkin graphs, 53
 A^{op} , 14
 $\text{AR}(\Pi_{n+3})$, AR quiver of Π_{n+3} , 292
 $\text{AR}(T)$, AR quiver of a triangulation, 291
 A^t , transpose of the matrix A , 151
 $\beta \hookrightarrow \alpha$, 209
 $B_Q(\alpha)$, Tits form, 50
 C^+, C^- , Coxeter functors, 66
 \mathbb{C} , complex numbers, 1
 c , Coxeter transformation, 67
 ${}^\perp C$, left orthogonal category, 260
 C^\perp , right orthogonal category, 260
 $C(f)$, cone of a morphism, 307
 χ , multiplicative character, 164, 176
 $\chi(V, W)$, Euler characteristic, 32
 $c_{\lambda, \mu}^\nu$, LR coefficient, 195
 $\text{Com}(\mathcal{A})$, category of complexes, 306
 $\text{Com}^b(\mathcal{A})$, category of bounded complexes, 306
 $\mathbb{C}Q$, path algebra, 11
 $\mathcal{C}(V)$, 26
 $\mathbb{C}[X]^G$, invariant ring, 154
 C_x^- , reflection functor, 60
 C_x^+ , reflection functor, 59
 $\text{Cyl}(f)$, cylinder of a morphism, 307
 D , duality functor, 103
 $\Delta(\mathbf{A}_n)$, associahedron, 288
 $\Delta(i, j, k)$, triangle with vertices i, j, k , 288
 Δ_Q , simplicial complex associated to Q , 297
 $\delta_{i,j}$, Kronecker symbol, 33
 $\underline{\dim}(V)$, dimension vector, 3
 $D(\lambda)$, Young diagram of λ , 193
 $\underline{d}_{T, [i,j]}$, dim. vector of triangulation and diagonal, 291
 d_W^V , 2, 28
 ϵ_i , dimension vector of simple S_i , 66
 $\text{Ext}_Q^i(V, W)$, 30
 e_k , elementary symmetric function, 154
 $\text{End}_A(P) = \text{Hom}_A(P, P)$ endomorphism ring, 44
 $E_Q(V, V')$, 295
 $\text{Ext}(V, W) = \text{Ext}^1(V, W)$, 27
 e_x , trivial path, 11
 $\text{ext}_Q(\alpha, \beta)$, generic ext, 209
 F_q , field with q elements, 143
 \mathbb{G}_a , additive group, 150
 GL_a , invertible $a \times a$ matrices, 6
 GL_α , 6

- $\mathrm{GL}_{\alpha, \sigma}$, 180
 \mathbb{G}_m , the multiplicative group, 150
 $\Gamma(Q)$, undirected graph of Q , 50
 $\mathrm{Grass}(k, m)$, Grassmann, 231

 $ha = h(a)$, head of a , 1
 $H^k(\mathcal{C})$, cohomology of a complex, 25
 $\mathrm{Hom}_Q(V, W)$, morphism space, 2
 $\mathrm{hom}_Q(\alpha, \beta)$, generic hom, 209
 $hp = h(p)$, head of a path, 11

 $\iota : G \rightarrow G$, inverse map, 150
 I_n , the $n \times n$ identity matrix, 151
 $\mathrm{Ind}(Q)$, 64
 in_x , 59
 I_x , injective representation, 22

 $K^b(\mathcal{A})$, homotopy category, 307

 $|\lambda|$, 193
 $\lambda(\alpha)$, 133
 $\ell(w)$, length of w , 136

 $m : G \times G \rightarrow G$, multiplication, 150
 $\mathrm{Mat}_{a, b}$, $a \times b$ matrices, 6
 $\mathrm{mod}\text{-}A$, 12

 \mathbb{N} , nonnegative integers, 3
 \mathcal{N} , Hilbert's nullcone, 163

 O_n , the orthogonal group, 151
 out_x , 61

 $\langle p \rangle$, 11
 Perm_d , 184
 PGL_{α} , 175
 $\mathrm{PGL}_{\alpha, \sigma}$, 180
 Φ , the Frobenius automorphism, 143
 $\tilde{\Phi}^+$, almost positive roots, 294
 Π_{n+3} , regular $(n+3)$ -gon, 288
 $\mathrm{Pol}(f)$, polarization of f , 185
 $\mathcal{P}_{u, v}(Q)$, preprojective algebra, 115
 $\mathbb{P}(V)^{\mathrm{ss}}$, semi-stable points, 168
 $\mathbb{P}(X)$, 167
 P_x , projective representation, 22
 $\pi : X \rightarrow X//G$, quotient, 162

 Q , quiver, 1
 Q_0 , vertices, 1
 Q_1 , arrows, 1
 $Q(T)$, quiver of a triangulation, 289

 \mathcal{R} , Reynolds operator, 155
 $\mathrm{rad}(A)$, Jacobson radical, 39

 $\mathrm{Rep}_{\alpha}(Q)$, representation space, 6
 $\mathrm{Rep}_{\alpha}(Q) //_{\sigma} \mathrm{GL}_{\alpha}$, 180
 $\mathrm{Rep}_{\alpha}(Q)^{\mathrm{ss}}$, $\mathrm{Rep}_{\alpha}(Q)^{\mathrm{s}}$, 179
 $\mathrm{Rep}(Q)$, category of representations, 3
 $\mathrm{Rep}(Q, J)$, 36

 $\sigma_x(\alpha)$, reflection of a vector, 58
 $\underline{\mathrm{sdim}}V$, signed dimension vector, 294
 $\tilde{\sigma}_x$, piecewise-linear reflection, 300
 $\mathrm{SI}(G, V)$, ring of semi-invariants, 165
 $\mathrm{SI}(Q, \beta)$, ring of semi-invariants, 166
 SL_{β} , 166
 S_n , symmetric group, 154
 $\mathrm{Spec}(R)$, spectrum of a ring R , 162
 $\sigma_x(Q)$, reflection of quiver, 58
 $\Sigma(Q, \beta)$, the cone of weights, 209
 S_{λ} , Schur functor, 193
 S_x , simple representation, 8

 $ta = t(a)$, tail of a , 1
 T_{α} , 175
 τ^+, τ^- , AR translation, 103
 $tp = t(p)$, tail of a path, 11
 $\mathrm{trace}(\cdot)$, trace of a linear map, 161
 $\mathrm{Tr}(V)$, transpose of a module V , 100

 U_n , unitary group, 156

 $V^{\otimes(d, e)}$, 184
 V^G , space of G -invariants, 153
 V^{s} , set of stable vectors, 169
 V^{ss} , set of semi-stable points, 167
 $V \oplus W$, direct sum, 4
 $V \perp W$, V left orthogonal to W , 260

 \mathcal{W} Weyl group, 66

 $X//G$, quotient variety, 162

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This book is an introduction to the representation theory of quivers and finite dimensional algebras. It gives a thorough and modern treatment of the algebraic approach based on Auslander-Reiten theory as well as the approach based on geometric invariant theory. The material in the opening chapters is developed starting slowly with topics such as homological algebra, Morita equivalence, and Gabriel's theorem. Next, the book presents Auslander-Reiten theory, including almost split sequences and the Auslander-Reiten transform, and gives a proof of Kac's generalization of Gabriel's theorem. Once this basic material is established, the book goes on with developing the geometric invariant theory of quiver representations. The book features the exposition of the saturation theorem for semi-invariants of quiver representations and its application to Littlewood-Richardson coefficients. In the final chapters, the book exposes tilting modules, exceptional sequences and a connection to cluster categories.

The book is suitable for a graduate course in quiver representations and has numerous exercises and examples throughout the text. The book will also be of use to experts in such areas as representation theory, invariant theory and algebraic geometry, who want to learn about applications of quiver representations to their fields.

ISBN 978-1-4704-2556-2



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