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An Introduction to Quiver Representations

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Preface

The idea of writing this book came from the collaboration of the authors. We started working on quiver representation when the first author visited Northeastern University as a Research Scholar in the academic year 1997/1998. We both came from other fields of algebra so we had to learn the subject from scratch. At the same time we were interested in connections of quiver representations with invariant theory, representations of algebraic groups, and algebraic geometry.

This experience made us realize that (except for Crawley-Boevey's notes [**18**] available online), there is no introductory text which would allow a person without any knowledge of Artin algebras to learn the subject quickly.

Later both authors taught courses on quiver representations at University of Michigan and Northeastern University, respectively. The second author also taught a two month course on quiver representations at Tor Vergata University in Rome in the spring of 2007.

We are grateful to Calin Chindris, Jiarui Fei, Ryan Kinser, Andrea Appel, Andrew Carroll, Sachin Gautam, Daniel Labardini, Kavita Sutar, Salvatore Stella, Riccardo Aragona, Ada Boralevi, Cristina DiTrapano, Luca Moci, and other students who took part in these courses.

The book is addressed to non-specialists, who want to learn the subject without going through the extended preparation in algebra, just starting from basic linear algebra. It turned out to be impossible to be completely elementary, so in some places we use some basic algebraic geometry (mainly the dimension counts).

We do not prove the results in full generality, working over the field $\mathbb{C}$ of complex numbers. We also work mostly with acyclic quivers. We only cover Auslander-Reiten duality in the case of hereditary algebras.

The book reflects our point of view, so the semi-invariants are covered in detail and we stress their role in our approach. Some of the results could be proved just using stability conditions without mentioning semi-invariants, but we find the combinatorics of the rings of semi-invariants quite fascinating.

Still many important topics are left out, for example, Ringel-Hall algebras and Nakajima quiver varieties.

In recent years the field developed very quickly. New concepts and connections emerged. We wanted this development to be reflected in the book, hence there are chapters on orthogonal categories, exceptional sequences and cluster categories.

We stress the connections of quiver representations with representations of algebraic groups and moduli problems.

The authors thank the National Science Foundations for its support during the writing of this book.

We discussed the subject of the book with many mathematicians, with some of whom we coauthored our papers. We benefited from discussions with Prakash Belkale, Calin Chindris, Bill Crawley-Boevey, Jose Antonio de la Pena, Jiarui Fei, Lutz Hille, Kiyoshi Igusa, Ryan Kinser, Mark Kleiner, Visu Makam, Kent Orr, Charles Paquette, Idun Reiten, Claus Ringel, Ralf Schiffler, Aidan Schofield, Gordana Todorov, Michel Van den Bergh, and Andrei Zelevinsky.

The second author also wants to thank Piotr Dowbor, Daniel Simson and Andrzej Skowroński whose questions about the invariant theory of quivers aroused his interest in the subject.

Notation

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This book is an introduction to the representation theory of quivers and finite dimensional algebras. It gives a thorough and modern treatment of the algebraic approach based on Auslander-Reiten theory as well as the approach based on geometric invariant theory. The material in the opening chapters is developed starting slowly with topics such as homological algebra, Morita equivalence, and Gabriel's theorem. Next, the book presents Auslander-Reiten theory, including almost split sequences and the Auslander-Reiten transform, and gives a proof of Kac's generalization of Gabriel's theorem. Once this basic material is established, the book goes on with developing the geometric invariant theory of quiver representations. The book features the exposition of the saturation theorem for semi-invariants of quiver representations and its application to Littlewood-Richardson coefficients. In the final chapters, the book exposes tilting modules, exceptional sequences and a connection to cluster categories.

The book is suitable for a graduate course in quiver representations and has numerous exercises and examples throughout the text. The book will also be of use to experts in such areas as representation theory, invariant theory and algebraic geometry, who want to learn about applications of quiver representations to their fields.

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