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Schur Functors and Schur Complexes

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INTRODUCTION

The study of Schur functors has a relatively long history. Its main impetus derived from representation theory, originally in characteristic zero. Over the years, however, with the development of modular representations, and algebraic geometry over fields of positive characteristic, the need for a theory of universal polynomial functors increased and, since the mid-1960s, approaches to a characteristic-free treatment of Schur functors have been developing (see, for instance, the recent book of Green [11] in which the treatments by Carter and Lusztig [6], Higman [12], and Towber [20], among others, are discussed). Our own interest in such a treatment was awakened by the work of Lascoux [14] on resolutions of determinantal ideals. Although his thesis treated only the characteristic zero case, it suggested that a general and elementary theory of Schur functors could be developed using only the rudiments of multilinear algebra (involving the Hopf algebra structures of the symmetric, exterior, and divided power algebras). Moreover, this elementary development admitted of a natural generalization to the idea of Schur complexes, whose usefulness was demonstrated, for instance, in our construction of a universal minimal

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resolution of the ideal of submaximal minors of the generic m-by-n matrix [2].

The classical definition of Schur functors using Young symmetrizers described in the Appendix is not suitable for fields of characteristic different from zero. We give a brief sketch of the way Young symmetrizers are circumvented by the use of exterior and symmetric powers. Let R be a commutative ring and let F be a free R-module. There is a natural injection of $\Lambda^p(F)$ into the pth tensor power $\otimes^p(F)$ (see I.2) and a natural surjection of $\otimes^p(F)$ onto $S_p(F)$ (see I.3). Now let $\lambda=(\lambda_1,\dots,\lambda_q)$ be a partition and $\tilde{\lambda}=(\tilde{\lambda}_1,\dots,\tilde{\lambda}_t)$ be its transpose (see II.1). The R-module $L_\lambda(F)$ is defined as the image of a composite map $d_\lambda(F)$:

$$\Lambda^{\lambda_1}F\otimes\cdots\otimes\Lambda^{\lambda_q}F\xrightarrow{\alpha_\lambda}\otimes^{|\lambda|}(F)\xrightarrow{\beta_\lambda}S_{\tilde{\lambda}_1}F\otimes\cdots\otimes S_{\tilde{\lambda}_t}F,$$

where α_λ is the tensor product of the injections $\Lambda^{\lambda_i}(F)\to\otimes^{\lambda_i}(F)$ and β_λ sends $f_1\otimes\cdots\otimes f_{|\lambda|}$ to

$$f_1f_{\lambda_1+1}f_{\lambda_1+\lambda_2+1}\cdots f_{\lambda_1+\cdots+\lambda_{q-1}+1}\otimes f_2f_{\lambda_1+2}f_{\lambda_1+\lambda_2+2}\cdots\otimes\cdots.$$

Since $d_\lambda(F)$ is a natural transformation, $L_\lambda(F)$ is a functor, called the Schur functor associated to λ. We prove in Chapter II that $L_\lambda(-)$ is a universally free functor, that is, $L_\lambda(F)$ is a free R-module and commutes with change of the base ring R. When λ is the partition (p), we have $L_\lambda(F)=\Lambda^p(F)$ and $L_{\tilde{\lambda}}(F)=S_p(F)$. If K is a field of characteristic zero, the modules $\{L_\lambda(K^n)\mid\lambda$ is a partition, $\lambda_1\leqslant n\}$ give a complete set of distinct irreducible polynomial representations of the general linear group $GL(n,K)$. It should be noted that the module $L_\lambda(\mathbb{Z}^n)$ is only one choice among many $\mathbb{Z}$-forms of the irreducible $GL(n,\mathbb{Q})$-module $L_\lambda(\mathbb{Q}^n)$, except when λ is of the form (p). To illuminate this remark, consider the case $\lambda=(1,\dots,1)$, where 1 repeats p-times, so that $L_\lambda(-)=S_p(-)$. The divided power functor $D_p(-)$ is the natural dual of $S_p(-)$ (see I.4) and gives another $\mathbb{Z}$-form: the $GL(n,\mathbb{Z})$-modules $D_p(\mathbb{Z}^n)$ and $S_p(\mathbb{Z}^n)$ are not isomorphic but the $GL(n,\mathbb{Q})$-modules $D_p(\mathbb{Z}^n)\otimes\mathbb{Q}$ and $S_p(\mathbb{Z}^n)\otimes\mathbb{Q}$ are both isomorphic to $S_p(\mathbb{Q}^n)$.

Chapter I is a review of background material on Hopf algebras which is utilized in the main body of the paper. Included are discussions of the relevant properties of the exterior, symmetric, and divided power algebras. Chapter II deals with the fundamental properties of Schur functors on free modules. After a discussion of partitions and (Young) diagrams, the Schur functor $L_\lambda(F)$ is defined in II.1 as the image of a natural transformation $d_\lambda(F)$: $\Lambda_\lambda(F)\to S_{\tilde{\lambda}}(F)$. More generally, given a pair of partitions $\mu\subseteq\lambda$, we define the (skew) Schur functor $L_{\lambda/\mu}(F)$ as the image of a natural map $d_{\lambda/\mu}(F)$: $\Lambda_{\lambda/\mu}(F)\to S_{\tilde{\lambda}/\tilde{\mu}}(F)$. When μ is the zero partition, $L_{\lambda/\mu}(F)=L_\lambda(F)$.

The coSchur functor $K_{\lambda/\mu}(F)$ is defined in a dual manner as the image of a map $d'_{\lambda/\mu}(F)$: $D_{\lambda/\mu}(F) \to \Lambda_{\tilde{\lambda}/\tilde{\mu}}(F)$.

In II.2 we prove that the Schur functors $L_{\lambda/\mu}(-)$ are universally free (polynomial) functors. The proof involves finding a "generators and relations" description of $L_{\lambda/\mu}(F)$ in terms of exterior powers and utilizing this description to construct a standard basis of $L_{\lambda/\mu}(F)$ from Young tableaux. The generators and relations developed here are used heavily throughout the paper, especially for the purpose of defining maps on Schur functors. In the case $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$, the Schur functor $L_{\lambda/\mu}(F)$ is described as the cokernel of a natural map $\square_{(\lambda_1,\lambda_2)/(\mu_1,\mu_2)}$:

$$\sum_{t=\mu_1-\mu_2+1}^{\lambda_2-\mu_2} \Lambda^{\lambda_1-\mu_1+t}(F) \otimes \Lambda^{\lambda_2-\mu_2-t}(F) \to \Lambda^{\lambda_1-\mu_1}(F) \otimes \Lambda^{\lambda_2-\mu_2}(F).$$

It should be noted that over a field of characteristic zero, only the term $t = \lambda_2 - \mu_2 + 1$ is necessary. As for partitions λ, μ of length >2, the relations for $L_{\lambda/\mu}(F)$ can be described in terms of the relations $\square_{(\lambda_i,\lambda_{i+2})/(\mu_i,\mu_{i+1})}$ given above (see II.2.10).

The proof of the universality of the coSchur functors $K_{\lambda/\mu}(F)$ is given in II.3 and proceeds along the same lines as the proof for Schur functors. The Schur and coSchur functors satisfy the duality $(L_{\lambda/\mu}(F))^* \cong K_{\tilde{\lambda}/\tilde{\mu}}(F^*)$ (see II.4.1). Over a field of characteristic zero, $K_{\tilde{\lambda}/\tilde{\mu}}(F)$ and $L_{\lambda/\mu}(F)$ are naturally isomorphic. The duality $\Lambda^p(F) = \Lambda^n(F) \otimes \Lambda^{n-p}(F^*)$, where $n = \text{rank}(F)$, is generalized in II.4.2 for all Schur functors. The remainder of II.4 is devoted to the proof of the decomposition $L_{\lambda/\mu}(F \oplus G) \cong \sum_{\mu \subseteq \sigma \subseteq \lambda} L_{\sigma/\mu}(F) \otimes L_{\lambda/\sigma}(G)$ which holds up to a natural filtration.

In Chapter III we prove the characteristic-free decompositions $S(F \otimes G) = \sum L_\lambda(F) \otimes L_\lambda(G)$ and $\Lambda(F \otimes G) = \sum L_\lambda(F) \otimes K_\lambda(G)$ which hold up to natural filtrations. These are the analogs of the Cauchy formulas for symmetric functions. The characteristic-free decomposition of $S(F \otimes G)$ was first proved by Doubilet *et al.* in [10]. Analogous decompositions can be found also in [1, 8, 19, 22, 23].

Chapter IV deals with the analog for Schur functors of the Littlewood–Richardson rule for the multiplication of Schur functions (see [17]). More precisely, we construct an explicit isomorphism between $L_\lambda(F) \otimes L_\mu(F)$ and the direct sum $\sum_\nu a^\nu_{\lambda\mu} L_\nu(F)$ over a field of characteristic zero, where the $a^\nu_{\lambda\mu}$ are the Littlewood–Richardson coefficients. This explicit isomorphism is convenient for calculating resolutions of generic ideals (e.g., Pfaffians, Plücker, determinantal) and modules in characteristic zero.

We begin Chapter V by defining the exterior and symmetric powers, $\Lambda^p\phi$ and $S_p\phi$, of a morphism ϕ: $G \to F$ of free modules over an arbitrary commutative ring R. $\Lambda^p(\phi)$ is the free chain complex

$$(0 \to D_p G \to \cdots \to \Lambda^{p-i} F \otimes D_i G \to \cdots \to \Lambda^p F \to 0)$$

and $S_p(\phi)$ is the free chain complex

$$(0 \to \Lambda^p G \to \cdots \to S_{p-i}F \otimes \Lambda^i G \to \cdots \to S_p F \to 0).$$

We then define the Schur complex $L_{\lambda/\mu}(\phi)$, in the same formal manner as Schur and coSchur functors, as the image of a natural chain map $d_{\lambda/\mu}(\phi)$: $\Lambda_{\lambda/\mu}(\phi) \to S_{\tilde{\lambda}/\tilde{\mu}}(\phi)$. Over fields of characteristic zero, Nielsen gives a definition of the Schur complex of a complex, using Young symmetrizers [18]. We prove that the Schur complex $L_{\lambda/\mu}(\phi)$ is a universal functor from maps of free R-modules to free chain complexes over R. This proof proceeds in the manner of the proofs in II.2 and II.3, that is, by the construction of a presentation of $L_{\lambda/\mu}(\phi)$ in terms of exterior powers of ϕ and the construction of a standard basis for $L_{\lambda/\mu}(\phi)$. The module of ith degree chains of the complex $L_{\lambda/\mu}(\phi)$ decomposes into the sum $\sum_{\mu \subseteq \sigma \subseteq \lambda, |\sigma| = i} K_{\sigma/\mu}(G) \otimes L_{\lambda/\sigma}(F)$ up to a natural filtration. Related characteristic-free constructions can be found in Stein's thesis [19].

In V.1 we also prove that the Schur complex $L_{\lambda/\mu}(\phi_1 \oplus \phi_2)$ decomposes, by a natural filtration of subcomplexes, into the sum $\sum_{\mu \subseteq \sigma \subseteq \lambda} L_{\sigma/\mu}(\phi_1) \otimes L_{\lambda/\sigma}(\phi_2)$ of tensor products of Schur complexes. In particular, if ϕ_1 is an isomorphism, the complexes $L_{\lambda/\mu}(\phi_1 \oplus \phi_2)$ and $L_{\lambda/\mu}(\phi_2)$ are homotopy equivalent. As a consequence of their universality and their decomposition properties, the Schur complexes have proved useful in the construction of characteristic-free minimal resolutions of generic ideals and modules. They are utilized in [2] to construct such resolutions of the ideal of submaximal minors and the powers of the ideal of maximal minors of a generic matrix.

In V.2 we define the Schur functor $L_{\lambda/\mu}(M)$ of an arbitrary module M over a commutative ring R, using generators and relations constructed from exterior powers of the module M. We prove that if $G \to^{\phi} F \to M \to 0$ is a finite free presentation of M, then the homology of the Schur complex $L_{\lambda/\mu}(\phi)$ in degree zero is $L_{\lambda/\mu}(M)$. Finally, we use Schur complexes to construct a universal minimal free resolution of the module $\Lambda^{m-n+2}(M)$, where M is the cokernel of a generic n-by-m matrix ($n \leqslant m$).

The Appendix contains a brief summary of results from the classical representation theory of the general linear group.

I. Background Material

I.1. *Hopf Algebras*

Definition I.1.1. Let R be a commutative ring. A graded R-coalgebra is a graded R-module $A = \sum_{i \geqslant 0} A_i$ together with a homogeneous comultiplication (or diagonalization) Δ: $A \to A \otimes A$ and a counit ε: $A \to R$

which satisfy properties dual to the properties satisfied by the multiplication $m\colon B \otimes B \to B$ and unit $\eta\colon R \to B$ of an (associative) R-algebra B. More precisely, the following diagrams are commutative:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ A & \xrightarrow{\Delta\otimes 1} & A\otimes A\otimes A \end{array} \qquad \begin{array}{ccccc} & & A & & \\ & \overset{\approx}{\swarrow} & \downarrow & \overset{\approx}{\searrow} & \\ R\otimes A & \xleftarrow{\epsilon\otimes 1} & A\otimes A & \xrightarrow{1\otimes\epsilon} & A\otimes R. \end{array}$$

If M, N are graded R-modules the twisting morphism $T\colon M \otimes N \to N \otimes M$ is the R-map defined by $T(x \otimes y) = (-1)^{ij} y \otimes x$ for $x \in M_i, y \in N_j$.

Definition I.1.2. By a (graded) R-Hopf algebra we shall mean a graded R-module A together with a multiplication $m\colon A \otimes A \to A$, a unit $\eta\colon R \to A$, a comultiplication $\Delta\colon A \to A \otimes A$ and a counit $\varepsilon\colon A \to R$ satisfying the two properties:

(1) (A, m, η) is a graded R-algebra, (A, Δ, ε) is a graded R-coalgebra, $\varepsilon\colon A \to R$ is a map of R-algebras, $\eta\colon R \to A$ is a map of R-coalgebras.

(2) m and Δ are compatible in the sense that the following diagram commutes:

$$\begin{array}{ccc} A\otimes A & \xrightarrow{m} A \xrightarrow{\Delta} & A\otimes A \\ {\scriptstyle\Delta\otimes\Delta}\downarrow & & \downarrow{\scriptstyle m\otimes m} \\ A\otimes A\otimes A\otimes A & \xrightarrow{1\otimes T\otimes 1} & A\otimes A\otimes A\otimes A. \end{array}$$

Observe that given condition (1), the second condition is equivalent to saying that $\Delta\colon A \to A \otimes A$ is a map of R-algebras or that $m\colon A \otimes A \to A$ is a map of R-coalgebras.

If, in addition, the diagrams

$$\begin{array}{ccc} A\otimes A & \xrightarrow{T} & A\otimes A \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array} \qquad \begin{array}{ccc} & A & \\ & {\scriptstyle\Delta}\swarrow \quad \searrow{\scriptstyle\Delta} & \\ A\otimes A & \xrightarrow[T]{} & A\otimes A \end{array}$$

commute, we say that A is a commutative (graded) R-Hopf algebra. In this section we will assume that all R-Hopf algebras are commutative, connected ($A_0 = R$), and free of finite type (A_i is a finitely generated free R-module for every i).

DEFINITION I.1.3. Let A, B be R-Hopf algebras and $\alpha: A \to B$ be a map of R-modules. We say α is an R-Hopf algebra map if $m_B \circ (\alpha \otimes \alpha) = \alpha \circ m_A$ and $\Delta_B \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_A$.

DEFINITION I.1.4. The tensor product $A \otimes B$ of graded R-Hopf algebras A and B is defined in the customary manner by taking the underlying module to be the graded R-module $A \otimes_R B$ and setting $m_{A \otimes B}, \Delta_{A \otimes B}$ to be the compositions

$$A \otimes B \otimes A \otimes B \xrightarrow{1 \otimes T \otimes 1} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B,$$

$$A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes T \otimes 1} A \otimes B \otimes A \otimes B,$$

respectively. This tensor product is the coproduct in the category of graded R-Hopf algebras.

DEFINITION I.1.5. Let $A = \sum_{i \geqslant 0} A_i$ be a graded R-Hopf algebra. We let A^* denote the graded dual $\sum_{i \geqslant 0} A_i^*$, where $A_i^* = \operatorname{Hom}_R(A_i, R)$. There is a canonical R-Hopf algebra structure on A^*: $m_{A^*} = (\Delta_A)^*$, $\Delta_{A^*} = (m_A)^*$. If $a \in A$, $b \in A^*$, we write $\langle b, a \rangle$ for the image of $b \otimes a$ under the canonical pairing $\langle\, , \rangle$: $A^* \otimes A \to R$. If A, B are graded R-Hopf algebras, there is a canonical pairing $A^* \otimes B^* \otimes A \otimes B \to R$ defined by the composition $A^* \otimes B^* \otimes A \otimes B \to^{1 \otimes T \otimes 1} A^* \otimes A \otimes B^* \otimes B \to^{\langle , \rangle \otimes \langle , \rangle} R \otimes R \cong R$. This pairing gives a natural map $A^* \otimes B^* \to (A \otimes B)^*$ of graded R-Hopf algebras. Since we are assuming A, B to be free of finite type this map is an isomorphism.

DEFINITION I.1.6. Let $\alpha: B \to A^*$ be a map of R-Hopf algebras. The pairing $\langle\, , \rangle_\alpha : B \otimes A \to R$ induced by α is the composition $B \otimes A \to^{\alpha \otimes 1} A^* \otimes A \to^{\langle , \rangle} R$. The B-module structure $\eta_\alpha : B \otimes A \to A$ on A induced by α is the composition

$$B \otimes A \xrightarrow{1 \otimes \Delta} B \otimes A \otimes A \xrightarrow{\langle , \rangle_\alpha \otimes 1} R \otimes A = A.$$

We write $b(a)$ for $n_\alpha(b \otimes a)$. It is easy to see if $\Delta(a) = \sum a_i \otimes a_i'$ then $b(a) = \sum \langle b, a_i \rangle_\alpha a_i'$.

DEFINITION I.1.7. Let A be an R-Hopf algebra. Define an R-map $\square_A$: $A \otimes A \to A \otimes A$ to be the composition $A \otimes A \to^{\Delta \otimes 1} A \otimes A \otimes A \to^{1 \otimes m} A \otimes A$. Similarly, define $\tilde{\square}_A : A \otimes A \to A \otimes A$ to be the composition $A \otimes A \to^{1 \otimes \Delta} A \otimes A \otimes A \to^{m \otimes 1} A \otimes A$.

PROPOSITION I.1.8. Let $B \to A^*$ be a map of R-Hopf algebras and let

$X_1, X_2 \in B$, $Y_1, Y_2 \in A$. Then $\langle \square_B(X_1 \otimes X_2),\ Y_1 \otimes Y_2 \rangle_{\alpha \otimes \alpha} = \langle X_1 \otimes X_2, \tilde{\square}_A(Y_1 \otimes Y_2) \rangle_{\alpha \otimes \alpha}$.

Proof. Since α is a map of R-Hopf algebras we may assume $B = A^*$. But now the proposition becomes the statement that $\square_{A^*} = (\tilde{\square}_A)^*$ which is immediate from the definitions.

I.2. *The Exterior Algebra*

The exterior algebra of a finitely generated free R-module F is the free graded commutative R-algebra generated by elements of F in degree one and is denoted $\Lambda F = \sum \Lambda^r F$. It is constructed as the quotient $T(F)/\mathscr{A}$, where $T(F) = \sum_{r \geqslant 0} T_r(F)$ is the tensor algebra on F and $\mathscr{A} = \sum \mathscr{A}_r$ is the two-sided homogeneous ideal of $T(F)$ generated by elements of the type $f \otimes f$, where $f \in F$. The rth degree component $\Lambda^r F$ is $T_r(F)/\mathscr{A}_r$. Since $\Lambda^1 F = F$ the canonical projection $T_r(F) \to \Lambda^r F$ can be viewed as the component $F \otimes \cdots \otimes F \to \Lambda^r F$ of r-fold multiplication in ΛF. The diagonal map $F \to F \oplus F$ $(f \mapsto (f, f))$ induces an R-algebra map $\Lambda F \to \Lambda(F \oplus F) \cong \Lambda F \otimes \Lambda F$ which is the comultiplication Δ of the Hopf algebra ΛF with the counit being the projection $\Lambda F \to R$ into degree 0. Note that if $f \in F$, $\Delta(f) = f \otimes 1 + 1 \otimes f$. Since Δ is an algebra map we have $\Delta(f_1 \wedge \cdots \wedge f_r) = \sum_{0 \leqslant s \leqslant r} \sum_\sigma \operatorname{sgn}(\sigma) f_{\sigma(1)} \wedge \cdots \wedge f_{\sigma(s)} \otimes f_{\sigma(s+1)} \wedge \cdots \wedge f_{\sigma(r)}$, where the second sum is over all permutations σ of $\{1, \ldots, r\}$ such that $\sigma(1) < \cdots < \sigma(s)$ and $\sigma(s+1) < \cdots < \sigma(r)$. The component $\Lambda^r F \to F \otimes \cdots \otimes F$ of r-fold comultiplication is the antisymmetrization map $f_1 \wedge \cdots \wedge f_r \mapsto \sum \operatorname{sgn}(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(r)}$ and it is a split monomorphism (over R). If the characteristic of R is not 2, the image is the module of antisymmetric r-tensors. If $F^* = \operatorname{Hom}(F, R)$ then the natural map $\Lambda(F^*) \to (\Lambda F)^*$ is an isomorphism of R-Hopf algebras. Finally $\Lambda(-)$ is an additive functor from the category of finitely generated free R-modules to the category of graded R-Hopf algebras.

I.3. *The Symmetric Algebra*

The symmetric algebra of a finitely generated free R-module F is the free graded commutative R-algebra generated by elements of F in degree 2 and is denoted $SF = \sum_{r \geqslant 0} S_r F$, where we write $S_r F$ for the elements of degree $2r$. SF is constructed as the quotient $T(F)/\mathscr{C}$, where $\mathscr{C}$ is the two-sided homogeneous ideal of the tensor algebra $T(F)$ generated by elements of the form $f_1 \otimes f_2 - f_2 \otimes f_1$, where $f_1, f_2 \in F$. Since $S_1 F = F$ the canonical projection $T_r(F) \to S_r F$ is the component $F \otimes \cdots \otimes F \to S_r F$ of r-fold multiplication in SF. The diagonal map $F \to F \oplus F$ induces an R-algebra map $SF \to S(F \oplus F) = SF \otimes SF$, which is the comultiplication of the Hopf algebra SF with the counit being the projection $SF \to R$ into degree zero. If $f \in F$, $\Delta(f) = f \otimes 1 + 1 \otimes f$. Since Δ is an algebra map we have

$$\Delta(f_1^{\alpha_1} \cdots f_t^{\alpha_t}) = \sum_{0 \leqslant \beta_i \leqslant \alpha_i} \binom{\alpha}{\beta} f_1^{\beta_1} \cdots f_t^{\beta_t} \otimes f_1^{\alpha_1 - \beta_1} \cdots f_t^{\alpha_t - \beta_t},$$

where

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} \quad \text{and} \quad \binom{\alpha_i}{\beta_i} = \frac{\alpha_i!}{\beta_i!\,(\alpha_i - \beta_i)!}.$$

The component $S_rF \to F \otimes S_{r-1}F$, which looks like $f_1^{\alpha_1} \cdots f_t^{\alpha_t} \to \sum_{i=1}^t f_i \otimes (\partial/\partial f_i)(f_1^{\alpha_1} \cdots f_t^{\alpha_t})$, is the classical (partial) polarization map. The component $S_rF \to F \otimes \cdots \otimes F$ of r-fold comultiplication is the complete polarization map. When R contains a field of characteristic zero the image is the module of symmetric r-tensors. $S(-)$ is an additive functor from the category of finitely generated free R-modules to the category of graded R-Hopf algebras.

There is a short exact sequence $0 \to \Lambda^2F \to F \otimes F \to S_2F \to 0$. More generally, S_rF is the cokernel of the map

$$\sum_{i=1}^{r-1} F \otimes \cdots \otimes \Lambda_i^2 F \otimes \cdots \otimes F \xrightarrow{1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1} F \otimes \cdots \otimes F.$$

This kind of presentation will be generalized for arbitrary Schur functors (see II.2.11).

I.4. *The Divided Power Algebra*

The divided power algebra DF of a finitely generated free R-module F can be defined in various equivalent ways. The quickest of these is to define DF as the graded dual of the Hopf algebra $S(F^*)$. To motivate this definition we will show that this is essentially the same as taking DF to be the algebra $\Phi(F)$ of partial differential operators on polynomial functions from F to R. The algebra of polynomial functions from F to R is canonically identified with $S(F^*)$. A differential operator of degree r is a homogeneous R-linear map $\phi: S(F^*) \to S(F^*)$ of degree $-r$ which makes the following diagram commute for every $p \geqslant r$:

$$\begin{array}{ccc} S_pF^* & \xrightarrow{\phi} & S_{p-r}F^* \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\cong} \\ S_rF^* \otimes S_{p-r}F^* & \xrightarrow{\phi \otimes 1} & R \otimes S_{p-r}F^*. \end{array}$$

The set of differential operators of degree r is an R-submodule of $\mathrm{End}_R(SF^*) = \mathrm{Hom}_R(SF^*, SF^*)$, denoted by $\Phi_r(F)$. It is easy to check that $\Phi(F) = \sum \Phi_r(F)$ is a commutative graded subalgebra of the endomorphism R-algebra $\mathrm{End}_R(SF^*)$.

Let $\xi_1,..., \xi_n$ be a basis for F and let $x_1,..., x_n$ be the dual basis for F^*. We identify SF^* with the polynomial ring $R[x_1,..., x_n]$ in n variables. If $\alpha = (\alpha_1,..., \alpha_n)$ is a sequence of nonnegative integers, let D^α denote the partial differential operators $\partial^{|\alpha|}/(\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n})$. When R contains a field of characteristic zero, the D^α form an R-basis of $\Phi(F)$ [or, equivalently, $\Phi(F) \cong S(F)$]. This, however, is not true in general. Let $D^{(\alpha)}$ be the differential operator defined by $D^{(\alpha)}\{x_1^{\beta_1} \cdots x_n^{\beta_n}\} = \binom{\beta}{\alpha} x_1^{\beta_1-\alpha_1} \cdots x_n^{\beta_n-\alpha_n}$, where $\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}$. Note that $(\alpha_1! \cdots \alpha_n!)\, D^{(\alpha)} = D^\alpha$. The $D^{(\alpha)}$ form an R-basis of $\Phi(F)$ for an arbitrary R. Observe that $D^{(\alpha)} \cdot D^{(\beta)} = \binom{\alpha+\beta}{\beta} D^{(\alpha+\beta)}$.

Let DF denote the graded dual of the R-Hopf algebra $S(F^*)$. Then $DF = \sum D_r F$, where $D_r F = \mathrm{Hom}_R(S_r(F^*), R)$. We define a natural map $\lambda_r \colon \Phi_r(F) \to D_r F$ to be the composition

$$\Phi_r(F) \hookrightarrow \mathrm{Hom}_R(SF^*, SF^*) \xrightarrow{\mathrm{Hom}(i,\pi)} \mathrm{Hom}(S_m(F^*), R),$$

where i is the inclusion $S_m F^* \hookrightarrow SF^*$ and π is the projection $SF^* \to R$ into degree zero (i.e., the counit of SF^*). It is easy to check that $\lambda = \sum \lambda_m$ defines an isomorphism $\Phi(F) \to DF$ of R-algebras with $\{\lambda(D^{(\alpha)})\}$ being the basis dual to the basis $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n}\}$ of SF^*. In particular, $\lambda(\partial/\partial x_i) = \xi_i \in F (\cong F^{**} = D_1 F)$. Let $\xi_i^{(p)} = \lambda(D^{(\rho)})$, where $\rho = (0,..., p,...; 0)$ with p in the ith position. Then $p!\, \xi_i^{(p)} = \xi_i^p$ and $\lambda(D^{(\alpha)}) = \xi_1^{(\alpha_1)} \cdots \xi_n^{(\alpha_n)}$. This leads to a third definition of the divided power algebra.

The divided power algebra $DF = \sum D_i F$ can be defined as the graded commutative algebra generated by elements $f^{(i)}$ in degree $2i$, where $f \in F$ and i is a nonnegative integer, satisfying the following conditions:

(0) $D_0 F = R, D_1 F = F$;

(1) $f^{(0)} = 1, f^{(1)} = f, f^{(i)} \in D_i F$ for $f \in F$;

(2) $f^{(p)} f^{(q)} = \binom{p+q}{q} f^{(p+q)}$ for $f \in F$;

(3) $(f+g)^{(p)} = \sum_{k=0}^{p} f^{(p-k)} g^{(k)}$ for $f, g \in F$;

(4) $(fg)^{(p)} = f^p g^{(p)}$ for $f, g \in F$;

(5) $(f^{(p)})^{(q)} = [p, q] f^{(pq)}$ for $f \in F$ where $[p, q] = (pq)!/q!\, p^q!$.

As with the symmetric algebra we write $D_i F$ for the elements of degree $2i$. If $\xi_1,..., \xi_n$ is a basis for F then the set $\{\xi_1^{(\alpha_1)} \cdots \xi_n^{(\alpha_n)} \mid \alpha_1 + \cdots + \alpha_n = p\}$ is a basis for $D_p F$ and it is dual to the basis $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_1 + \cdots + \alpha_n = p\}$ of $S_p(F^*)$, where $x_1,..., x_n$ is the basis of F^* dual to $\xi_1,..., \xi_n$. DF has a graded

R-Hopf algebra structure as the graded dual of $S(F^*)$. It is easy to check that $\Delta_{DF}(f) = f \otimes 1 + 1 \otimes f$ for $f \in F$. Since $m_{SF^*}: SF^* \otimes SF^* \to SF^*$ is a map of coalgebras, $\Delta_{DF}: DF \to DF \otimes DF$ is a map of algebras. It follows that $\Delta_{DF}(f_1^{(\alpha_1)} \cdots f_t^{(\alpha_t)}) = \sum_{0 \leqslant \beta_i \leqslant \alpha_i} f_1^{(\beta_1)} \cdots f_t^{(\beta_t)} \otimes f_1^{(\alpha_1 - \beta_1)} \cdots f_t^{(\alpha_n - \beta_t)}$. The component $D_r F \to F \otimes \cdots \otimes F$ of r-fold diagonalization is a split monomorphism (over R) and its image is the module of symmetric r-tensors. There is a short exact sequence $0 \to D_2 F \to^{\Delta} F \otimes F \to^{m} \Lambda^2 F \to 0$. More generally, $\Lambda^r F$ is the cokernel of the map

$$\sum_{i=1}^{r-1} F \otimes \cdots \otimes D_{2_i} F \otimes \cdots \otimes F \xrightarrow{1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1} F \otimes \cdots \otimes F.$$

This kind of presentation will be generalized for arbitrary coSchur functors (see II.3.11).

There is a useful multiplication on DF which is unnatural in that it depends on a choice of basis for F. Let $\xi_1, ..., \xi_n$ be a basis for F. We define a product on DF, called integration with respect to the basis $\xi_1, ..., \xi_n$ of F and denoted by $\cup$, by taking $(\xi_1^{(\alpha_1)} \cdots \xi_n^{(\alpha_n)}) \cup (\xi_1^{(\beta_1)} \cdots \xi_n^{(\beta_n)}) = \xi_1^{(\alpha_1 + \beta_1)} \cdots \xi_n^{(\alpha_n + \beta_n)}$. The reason for the name "integration" is the fact that $\xi_1 \cup (\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}) = (1/(\alpha_1 + 1))\, \xi_1^{(\alpha_1 + 1)} \xi_2^{\alpha_2} \cdots \beta_n^{\alpha_n}$. It is easy to check that $\cup$ makes DF a commutative ring, isomorphic in fact to SF.

I.5. *Extended Hopf Algebras*

Let $\bar{R}$ be an R-algebra, possibly noncommutative. If M is an R-module we let $\bar{M}$ denote the two-sided $\bar{R}$-module $\bar{R} \otimes_R M$ with the customary $\bar{R}$ actions. We identify $\bar{M} \otimes_{\bar{R}} \bar{N}$ with $\bar{R} \otimes_R M \otimes_R N$ via the canonical isomorphism. By an extended $\bar{R}$-Hopf algebra we mean an $\bar{R}$-module $\bar{A}$, where A is an R-Hopf algebra, together with a multiplication $m_{\bar{A}} = 1 \otimes m_A: \bar{R} \otimes A \otimes A \to \bar{R} \otimes A$ and a comultiplication $\Delta_{\bar{A}} = 1 \otimes \Delta_A: \bar{R} \otimes A \to \bar{R} \otimes A \otimes A$. We also let $\bar{T}: \bar{A} \otimes_{\bar{R}} \bar{B} \to \bar{B} \otimes \bar{A}$ be the map $1 \otimes T: \bar{R} \otimes A \otimes B \to \bar{R} \otimes B \otimes A$. With these definitions we can formally extend Definitions I.1.2 through I.1.7 and Proposition I.1.8 to extended $\bar{R}$-Hopf algebras. If $\bar{F}$ is a free $\bar{R}$-module (where F is a free R-module), we let $\Lambda_{\bar{R}} \bar{F} = (\Lambda_R \bar{F})^-$, $S_{\bar{R}} \bar{F} = (S_R F)^-$, $(D_{\bar{R}} \bar{F}) = (D_R F)^-$.

I.6. *Bigraded Hopf Algebras*

The concept of graded R-Hopf algebras discussed in Section I.1 can be generalized to that of bigraded R-Hopf algebras by working in the category of bigraded R-modules and defining a twisting morphism. Let $M = \sum M_{i_1, i_2}$, $N = \sum N_{i_1, i_2}$ be bigraded R-modules. We take the twisting morphism $\dot{T}: M \otimes N \to N \otimes M$ in the category of bigraded R-modules to be the R-map given by $\dot{T}(x \otimes y) = (-1)^{(i_1 j_1 + i_1 j_2 + i_2 j_1)} y \otimes x$ for $x \in M_{i_1, i_2}$,

$y \in N_{j_1, j_2}$. Definitions I.1.1 through I.1.7, as well as Proposition I.1.8, can all be repeated in the category of bigraded R-modules with the twisting morphism $\dot{T}$.

As an example of a bigraded R-Hopf algebra, consider the antisymmetric tensor product $\Lambda F \dot{\otimes} DG$ of the graded R-Hopf algebras ΛF and DG, where F, G are finitely generated free R-modules, and view the elements of $D_i G$ as elements of degree i for the purpose of the tensor product. For convenience let $A = \Lambda F$, $B = DG$, $A_i = \Lambda^i F, B_i = D_i G$. The underlying R-module structure of $A \dot{\otimes} B$ is $A \otimes_R B$ with the bigrading $\sum A_{i_1} \otimes B_{i_2}$. There is a multiplication $m_{A \dot{\otimes} B}$: $(A \dot{\otimes} B) \otimes (A \dot{\otimes} B) \to A \dot{\otimes} B$ which is the composition $A \otimes B \otimes A \otimes B \to^{1 \otimes T \otimes 1} A \otimes A \otimes B \otimes B \to^{m_A \otimes m_B} A \otimes B$ and a comultiplication $\Delta_{A \dot{\otimes} B}: A \dot{\otimes} B \to (A \dot{\otimes} B) \otimes (A \dot{\otimes} B)$ which is the composition $A \otimes B \to^{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \to^{1 \otimes T \otimes 1} A \otimes B \otimes A \otimes B$, where T is the twisting morphism given by $T(x \otimes y) = (-1)^{ij} y \otimes x$ for $x \in A_i$, $y \in B_j$. As an illustration, let $x_t \otimes y_t \in A_{i_t} \otimes B_{j_t}$ $(t = 1, 2)$. Then $m_{A \otimes B}(x_1 \otimes y_1 \otimes x_2 \otimes y_2) = (-1)^{i_2 \cdot j_1} x_1 \cdot x_2 \otimes y_1 \cdot y_2$. It is easy to check that $A \dot{\otimes} B$ is a commutative bigraded R-Hopf algebra. For example, checking that $m_{A \dot{\otimes} B}$ is "commutative" is proving that the diagram

$$\begin{array}{ccc} (A \dot{\otimes} B) \otimes (A \dot{\otimes} B) & \xrightarrow{\dot{T}} & (A \dot{\otimes} B) \otimes (A \dot{\otimes} B) \\ & {\scriptstyle m_{A \dot{\otimes} B}} \searrow \quad \swarrow {\scriptstyle m_{A \dot{\otimes} B}} & \\ & A \dot{\otimes} B & \end{array}$$

commutes. Observe that we could have taken $A = \sum A_i$ to be any graded commutative R-Hopf algebra and $B = \sum B_i$ any graded R-module which has a graded R-Hopf algebra structure if the grading is altered so that B_i is of degree $2i$.

II. Schur Functors

II.1. *Definitions*

We will denote by $\mathbb{N}$ the set of natural numbers (i.e., nonnegative integers), and by $\mathbb{N}^\infty$ the set of sequences of elements of $\mathbb{N}$ of finite support (i.e., sequences in which all but a finite number of terms are zero). If $\mathbb{N}^p$ denotes the set of p-tuples of elements of $\mathbb{N}$, then $\mathbb{N}^p$ may, as usual, be identified with a subset of $\mathbb{N}^\infty$ by extending any p-tuple $(\lambda_1, \ldots, \lambda_p)$ by zeroes. Thus $\mathbb{N}^\infty = \bigcup_{p \geqslant 0} \mathbb{N}^p$, and we will not distinguish between the element $(\lambda_1, \ldots, \lambda_p)$ of $\mathbb{N}^p$ and the element $(\lambda_1, \lambda_2, \ldots, \lambda_p, 0, \ldots)$ in $\mathbb{N}^\infty$.

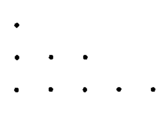
If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is an element of $\mathbb{N}^\infty$, we define the *conjugate* (or *transpose*) $\tilde{\lambda}$ of λ to be the element $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ of $\mathbb{N}^\infty$ where $\tilde{\lambda}_j$ is the number of terms of λ which are greater than or equal to j. For example, if $\lambda = (1, 0, 0, 3, 5, 4, 0, 0,...)$, then $\tilde{\lambda} = (4, 3, 3, 2, 1, 0, 0,...)$. Applying the same procedure, we see that $\tilde{\tilde{\lambda}} = (5, 4, 3, 1, 0, 0,...)$. In general, the conjugate $\tilde{\lambda}$ of any element $\lambda \in \mathbb{N}^\infty$ has the property that $\tilde{\lambda}_1 \geqslant \tilde{\lambda}_2 \geqslant \cdots$ and the element $\tilde{\tilde{\lambda}}$ is the sequence λ rearranged in decreasing order. Thus, if λ is a non-increasing sequence, $\lambda = \tilde{\tilde{\lambda}}$.

Definition II.1.1. A *partition* is an element $\lambda = (\lambda_1, \lambda_2,...)$ of $\mathbb{N}^\infty$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. The *weight* of the partition λ, denoted by $|\lambda|$, is the sum $\sum \lambda_i$. If $|\lambda| = n$, λ is said to be a *partition of n*. The number of non-zero terms of λ is called the *length* of λ.

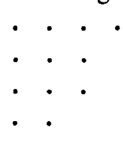
The observations above tell us that conjugation is an involution on the set of partitions.

Definition II.1.2. The *diagram* (or *shape*) of an element $\lambda \in \mathbb{N}^\infty$ is the set of ordered pairs (i,j) in $\mathbb{N}^2$ with $i \geqslant 1$ and $i \leqslant j \leqslant \lambda_i$, and is denoted by Δ_λ. Here we are adopting the convention that is used with matrices (and as described in [17]), namely, that the row index i increases as one goes downward, and the column index j increases from left to right. (For illuminating comments on this convention, the reader is referred to the footnote on page 2 of [17].)

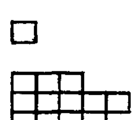
Since, for an arbitrary $\lambda \in \mathbb{N}^\infty$, λ_i is most often zero, the ith row is most often empty. The diagram for $\lambda = (1, 0, 0, 3, 5, 4, 0, 0,...)$, for example, looks like:



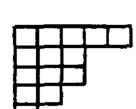
where rows 2 and 3, as well as all the rows below row 6, are empty. If λ is a partition, however, we have no intermediate gaps in the diagram. For instance, if $\lambda = (4, 3, 3, 2, 1)$, then its diagram looks like:



For convenience, the dots of a diagram are often replaced by squares. If we do that, the above schemes are represented as:



and



Using the diagrams, one can see that if λ is a partition, then $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots)$ is the partition whose jth term $\tilde{\lambda}_j$ is the number of squares in the jth column of the diagram of λ, where we count the columns from left to right. It is therefore clear that $|\lambda| = |\tilde{\lambda}|$ (i.e., conjugation is a weight-preserving involution on the set of partitions).

It should be pointed out that there is only one partition of weight zero, namely, $(0, 0, 0, \ldots)$, and this is of course called the *zero partition*, 0. Clearly, $\tilde{0} = 0$, and the diagram of 0 is empty.

If F is a free module over a commutative ring R, and $\lambda = (\lambda_1, \ldots, \lambda_q)$ is in $\mathbb{N}^\infty$, we use the following notation:

$$\Lambda_\lambda F = \Lambda^{\lambda_1} F \otimes_R \cdots \otimes_R \Lambda^{\lambda_q} F;$$

$$S_\lambda F = S_{\lambda_1} F \otimes_R \cdots \otimes_R S_{\lambda_q} F;$$

$$D_\lambda F = D_{\lambda_1} F \otimes_R \cdots \otimes_R D_{\lambda_q} F,$$

where Λ, S and D denote the exterior, symmetric and divided powers. Our aim, when λ is a partition, is to define a map

$$d_\lambda(F): \Lambda_\lambda F \to S_{\tilde{\lambda}} F,$$

associated to the partition λ and the free module F. In order to do this we introduce some additional notation.

If λ is a partition, we have its conjugate $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_t)$. Let λ' be the conjugate of $(\tilde{\lambda}_2, \ldots, \tilde{\lambda}_t)$. Then $\tilde{\lambda}' = (\tilde{\lambda}_2, \ldots, \tilde{\lambda}_t)$ and λ' is easily seen to be the partition whose diagram is obtained from that of λ by lopping off the first (left-most) column. Thus $|\lambda'| = |\lambda| - q$, where q is the length of λ, and if λ is not the zero partition, then $q > 0$, and $|\lambda'| < |\lambda|$.

We shall now define the map $d_\lambda(F)$ recursively with respect to the weight of λ. If λ has weight 0, then λ is the zero partition and $\Lambda_\lambda F = S_\lambda F = R$. We define the map $d_0(F): \Lambda_0 F \to S_0 F$ to be the identity map on R. Now suppose the weight of λ is positive and $d_\mu(F)$ has been defined for all partitions μ of weight less than $|\lambda|$. Consider the map

$$\delta_\lambda : \Lambda_\lambda F \to S_{\tilde{\lambda}_1} F \otimes \Lambda_{\lambda'} F$$

defined as the following composition of maps:

$$\Lambda_\lambda F = \Lambda^{\lambda_1} F \otimes \cdots \otimes \Lambda^{\lambda_q} F \xrightarrow{\Delta \otimes \cdots \otimes \Delta} F \otimes \Lambda^{\lambda_1 - 1} F \otimes F \otimes \Lambda^{\lambda_2 - 1} F \otimes \cdots$$

$$\otimes F \otimes \Lambda^{\lambda_q - 1} F \xrightarrow{\sigma} F \otimes F \otimes \cdots$$

$$\otimes F \otimes \Lambda_{\lambda'} F \xrightarrow{m \otimes 1} S_{\tilde{\lambda}_1} F \otimes \Lambda_{\lambda'} F.$$

The maps $\Delta: \Lambda^{\lambda_i}F \to F \otimes \Lambda^{\lambda_i - 1}F$ are the appropriate components of the diaonal map on ΛF, σ is the isomorphism commuting the F's past the $\Lambda^{\lambda_i - 1}F$ terms, and m is the multiplication from $F \otimes \cdots \otimes F$ to $S_{\tilde{\lambda}_1}F$ (remember that $\tilde{\lambda}_1 = q$).

Since we are supposing the map $d_{\lambda'}(F)$ to be already defined, we may compose the map δ_λ with the map $1 \otimes d_{\lambda'}(F)$: $S_{\tilde{\lambda}_1}F \otimes \Lambda_{\lambda'}F \to S_{\tilde{\lambda}_1}F \otimes S_{\tilde{\lambda}'}F$ giving us the map

$$\Lambda_\lambda F \to S_{\tilde{\lambda}_1}F \otimes S_{\tilde{\lambda}'}F.$$

But we have $\tilde{\lambda}' = (\tilde{\lambda}_2, \ldots, \tilde{\lambda}_t)$, so $S_{\tilde{\lambda}_1}F \otimes S_{\tilde{\lambda}'}F = S_{\tilde{\lambda}}F$ and the above map is, by definition, the map

$$d_\lambda(F): \Lambda_\lambda F \to S_{\tilde{\lambda}}F.$$

In an analogous way, we define the map

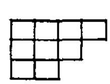
$$d'_\lambda(F): D_\lambda F \to \Lambda_{\tilde{\lambda}}F.$$

Definition II.1.3. The image of $d_\lambda(F)$ is defined to be the *Schur functor* of F with respect to the partition λ, and is denoted by $L_\lambda F$. The image of $d'_\lambda(F)$ is defined to be the *coSchur functor* of F with respect to the partition λ, and is denoted by $K_\lambda F$. $L_\lambda F$ $(K_\lambda F)$ is also called the *Schur (coSchur) functor of shape* λ.

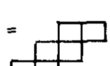
The foregoing material can now be generalized to the notions of skew-shapes and Schur (coSchur) functors of skew-shapes.

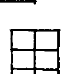
If $\lambda, \mu \in \mathbb{N}^\infty$, we say $\mu \subseteq \lambda$ if $\mu_i \leqslant \lambda_i$ for all $i \geqslant 1$. Clearly this is equivalent to stating that the diagram of μ is contained in that of λ. If $\mu \subseteq \lambda$, we define the *skew-shape*, $\Delta_{\lambda/\mu}$, to be $\Delta_\lambda - \Delta_\mu$. Since the sequence $\lambda - \mu = (\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots)$ is in $\mathbb{N}^\infty$, the reader may confuse $\Delta_{\lambda/\mu}$ with $\Delta_{\lambda - \mu}$. These two sets, however, are not the same, as the following example illustrates:

Let $\lambda = (4, 3, 2)$ and $\mu = (2, 1)$. Then

Δ_λ =


Δ_μ =


$\Delta_{\lambda/\mu}$ =


$\Delta_{\lambda-\mu}$ =


We shall never have any use for the diagram $\Delta_{\lambda-\mu}$, and it will be convenient to use the notation λ/μ for the sequence $\lambda-\mu$. Thus, λ/μ by itself will mean the sequence $\lambda-\mu$, but $\Delta_{\lambda/\mu}$ will signify the skew-shape $\Delta_\lambda - \Delta_\mu$. Of course, when $\mu=(0)$, $\lambda/\mu=\lambda$ and $\Delta_{\lambda/\mu}=\Delta_\lambda$.

If λ, μ are partitions with $\mu \subseteq \lambda$, we want to define maps

$$d_{\lambda/\mu}\colon \Lambda_{\lambda/\mu}F \to S_{\tilde{\lambda}/\tilde{\mu}}F,$$
$$d'_{\lambda/\mu}\colon D_{\lambda/\mu}F \to \Lambda_{\tilde{\lambda}/\tilde{\mu}}F$$

which reduce to the maps $d_\lambda$and d'_λ already defined, when $\mu=(0)$.

Note first that when the length of μ equals the length of λ, then $\lambda/\mu=\lambda'/\mu'$ and $\Delta_{\lambda/\mu}\approx\Delta_{\lambda'/\mu'}$, where $\lambda'(\mu')$ is the partition we described earlier, obtained from $\lambda(\mu)$ by lopping off the first column of dots from $\Delta_\lambda(\Delta_\mu)$. (We write $\Delta_{\lambda/\mu}\approx\Delta_{\lambda'/\mu'}$ to indicate that the two diagrams are isomorphic by horizontal translation.) In this case too, $X_{\lambda/\mu}F=X_{\lambda'/\mu'}F$ and $X_{\tilde{\lambda}/\tilde{\mu}}F=X_{\tilde{\lambda}'/\tilde{\mu}'}F$, where X stands for Λ, S or D.

Thus, to define the maps $d_{\lambda/\mu}$ and $d'_{\lambda/\mu}$, we may assume that the length of μ is $q' <$ length of $\lambda=q$, and that the maps have already been defined for all partitions of weight less than $|\lambda|$. (The weight zero case occurs only when $\lambda=\mu=(0)$, and in this case the maps have already been defined to be the identity.) We will define the map $d_{\mu/\lambda}$; the map $d'_{\lambda/\mu}$ is defined analogously.

Let τ be the partition $(\lambda_1,..., \lambda_{q'})$, and let ν be the partition $(\lambda_{q'+1},..., \lambda_q)$. Then we have the map

$$\Lambda_{\lambda/\mu}F=\Lambda_{\tau/\mu}F\otimes\Lambda_\nu F \xrightarrow{1\otimes\delta_\nu} \Lambda_{\tau/\mu}F\otimes S_{q-q'}F\otimes\Lambda_{\nu'}F,$$

where δ_ν was defined earlier. Note that $q-q'=\tilde{\lambda}_1-\tilde{\mu}_1$ and that $\tau/\mu=\tau'/\mu'$ (since τ and μ have the same length). Thus $\Lambda_{\tau/\mu}F\otimes\Lambda_{\nu'}F=\Lambda_{\lambda'/\mu'}F$ so that the above map is a map of $\Lambda_{\lambda/\mu}F$ into $S_{\tilde{\lambda}_1-\tilde{\mu}_1}F\otimes\Lambda_{\lambda'/\mu'}F$. Since $|\lambda'|<|\lambda|$, we already have the map

$$d_{\lambda'/\mu'}\colon\Lambda_{\lambda'/\mu'}F\to S_{\tilde{\lambda}_2-\tilde{\mu}_2}F\otimes\cdots\otimes S_{\tilde{\lambda}_t-\tilde{\mu}_t}F$$

so that the composition

$$\Lambda_{\lambda/\mu}F\xrightarrow{1\otimes\delta_\nu} S_{\tilde{\lambda}_1-\tilde{\mu}_1}F\otimes\Lambda_{\lambda'/\mu'}F\xrightarrow{1\otimes d_{\lambda'/\mu'}}$$

$$S_{\tilde{\lambda}_1-\tilde{\mu}_1}F\otimes S_{\tilde{\lambda}_2-\tilde{\mu}_2}F\otimes\cdots\otimes S_{\tilde{\lambda}_t-\tilde{\mu}_t}F=S_{\tilde{\lambda}/\tilde{\mu}}F$$

is our map $d_{\lambda/\mu}$. It is easy to see from this recursive definition that the map $d_{\lambda/\mu}$ coincides with the map d_λ when $\mu=(0)$.

A more direct way of defining the map $d_{\lambda/\mu}$ is by means of a t-by-t matrix (a_{ij}), called the Ferrers matrix associated to λ/μ, defined by $a_{ij} = 1$ for $\mu_i + 1 \leqslant j \leqslant \lambda_i$ and $a_{ij} = 0$ for $1 \leqslant j \leqslant \mu_i$ or $\lambda_{j+1} \leqslant j \leqslant t$ where $t = \lambda_1$.

For each $i = 1,..., q$, we have the iterated diagonal map

$$\Delta_i : \Lambda^{\lambda_i - \mu_i} F \to \Lambda^{a_{i1}} F \otimes \cdots \otimes \Lambda^{a_{it}} F = S_{a_{i1}} F \otimes \cdots \otimes S_{a_{it}} F.$$

Notice that $a_{1j} + a_{2j} + \cdots + a_{qj} = \tilde{\lambda}_j - \tilde{\mu}_j$ for $j = 1,..., t$. We therefore have the composition

$$\Lambda_{\lambda/\mu} F \xrightarrow{\Delta_1 \otimes \cdots \otimes \Delta_q} S_{a_{11}} F \otimes \cdots \otimes S_{a_{1t}} F \otimes \cdots S_{a_{q1}} F \otimes \cdots \otimes$$

$$S_{a_{qt}} F \xrightarrow{m_1 \otimes \cdots \otimes m_t} S_{\tilde{\lambda}_1 - \tilde{\mu}_1} F \otimes \cdots \otimes S_{\tilde{\lambda}_t - \tilde{\mu}_t} F = S_{\tilde{\lambda}/\tilde{\mu}} F,$$

where $m_j : S_{a_{1j}} F \otimes \cdots \otimes S_{a_{qj}} F \to S_{\tilde{\lambda}_j - \tilde{\mu}_j} F$ is the multiplication map for $j = 1,..., t$. This composition is the map $d_{\lambda/\mu}$.

Definition II.1.4. The image of $d_{\lambda/\mu} : \Lambda_{\lambda/\mu} F \to S_{\tilde{\lambda}/\tilde{\mu}} F$ is called the *Schur functor of F with respect to the skew-shape λ/μ*, and is denoted by $L_{\lambda/\mu} F$. The image of $D'_{\lambda/\mu} : D_{\lambda/\mu} F \to \Lambda_{\tilde{\lambda}/\tilde{\mu}} F$ is called the *coSchur functor of F with respect to the skew-shape λ/μ*, and is denoted by $K_{\lambda/\mu} F$.

We can generalize the above map in the following way. Let $\alpha = (a_{ij})$ be any $s \times t$ matrix of zeroes and ones with $p_i = \sum_j a_{ij}$ and $q_j = \sum_i a_{ij}$. Let $\Lambda_\alpha F = \Lambda^{p_1} F \otimes \cdots \otimes \Lambda^{p_s} F$, $S_{\tilde{\alpha}} F = S_{q_1} F \otimes \cdots \otimes S_{q_t} F$ ($\tilde{\alpha}$ = transpose of α). We define a natural map

$$d_\alpha : \Lambda_\alpha F \to S_{\tilde{\alpha}} F$$

as the composition

$$\Lambda_\alpha F \to \Lambda^{a_{11}} F \otimes \cdots \otimes \Lambda^{a_{st}} F = S_{a_{11}} F \otimes \cdots \otimes S_{a_{st}} F \to S_{\tilde{\alpha}} F$$

in the same manner as the map $d_{\lambda/\mu}$ above was defined.

Lemma II.1.5. *Let $\alpha = (a_{ij})$ be a matrix as above, and let $\sigma \in \mathrm{Sym}\{1,..., t\}$ (i.e., σ is a permutation of $\{1,..., t\}$). Define $\beta = (b_{ij})$ to be the matrix given by $b_{ij} = a_{i\sigma(j)}$. Then the following diagram is commutative (up to sign):*

$$\begin{array}{ccc} & \Lambda_\alpha F = \Lambda_\beta F & \\ {}^{d_\alpha}\swarrow & & \searrow^{d_\beta} \\ S_{\tilde{\alpha}} F & \longrightarrow & S_{\tilde{\beta}} F \end{array}$$

(where the bottom map is the natural isomorphism

$$s_{q_1}F \otimes \cdots \otimes S_{q_t}F \to S_{q_{\sigma(1)}}F \otimes \cdots \otimes S_{q_{\sigma(t)}}F$$

which permutes terms). Consequently $\operatorname{Im} d_\alpha \cong \operatorname{Im} d_\beta$.

LEMMA II.1.6. *Let* $\alpha = (a_{ij})$ *be a matrix as above, and let* $\sigma \in \operatorname{Sym}\{1,..., s\}$. *Let* $\beta = (b_{ij})$ *be defined by* $b_{ij} = a_{\sigma(i)j}$. *Then the following diagram is commutative:*

$$\begin{array}{ccc} \Lambda_\alpha F & \longrightarrow & \Lambda_\beta F \\ & {\scriptstyle d_\alpha}\searrow \quad \swarrow{\scriptstyle d_\beta} & \\ & S_{\tilde{\alpha}}F = S_{\tilde{\beta}}F & \end{array}$$

(*where the top map is the natural "twisting" isomorphism*

$$\Lambda^{p_1}F \otimes \cdots \otimes \Lambda^{p_s}F \to \Lambda^{p_{\sigma(1)}}F \otimes \cdots \otimes \Lambda^{p_{\sigma(s)}}F).$$

Consequently $\operatorname{Im} d_\alpha \cong \operatorname{Im} d_\beta$.

The proofs of these lemmas are simple applications of (co-)associativity and (co–)commutativity of these Hopf algebras.

II.2. *The Universal Freeness of Schur Functors*

We now proceed to a proof of the fact that the modules $L_{\lambda/\mu}F$ are universally free. We will do this by exhibiting a basis for these modules (showing that they are free) and by describing these modules as cokernels of maps between universally free modules.

If $\{x_1,..., x_n\}$ is a basis for the module F, and $I = \alpha_1 < \cdots < \alpha_s$ is a strictly increasing subset of $\{1,..., n\}$, we let x_I denote $x_{\alpha_1} \wedge \cdots \wedge x_{\alpha_s} \in \Lambda^s F$. Letting $\mu \subseteq \lambda$ be partitions, we see that the elements $x_{I_1} \otimes \cdots \otimes x_{I_q}$ form a basis of $\Lambda_{\lambda/\mu}F$, where I_i is a strictly increasing subset of $\{1,..., n\}$ having $\lambda_i - \mu_i$ elements, and q is the length of λ. Therefore the elements $d_{\lambda/\mu}(x_{I_1} \otimes \cdots \otimes x_{I_q})$ are a set of generators for $L_{\lambda/\mu}F$. We will show that a subset of these generators is actually a basis for $L_{\lambda/\mu}F$, and we now introduce some terminology to enable us to describe precisely which of these generators form the basis.

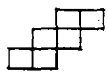
DEFINITION II.2.1. Let S be a totally ordered set, let λ and μ be partitions with $\mu \subseteq \lambda$, and let $\Delta_{\lambda/\mu}$ be the skew-shape associated to this pair. A *tableau of shape* λ/μ *with values in the set* S is a function from $\Delta_{\lambda/\mu}$ to S. The set of all such tableaux is denoted by $\operatorname{Tab}_{\lambda/\mu}(S)$.

In particular, we may choose S to be the ordered basis $\{x_1,..., x_n\}$ of our free module F. If $X = X_{I_1} \otimes \cdots \otimes X_{I_q}$ is a basis element of $\Lambda_{\lambda/\mu}F$, we have the tableau $T_X : \Delta_{\lambda/\mu} \to \{x_1,..., x_n\}$ defined as follows:

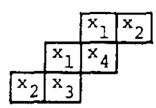
$$T_X(i,j) = x_{\alpha_{ij'}},$$

where $j' = j - \mu_i$ and where $I_i = (\alpha_{i1},..., \alpha_{iv_i})$ with $v_i = \lambda_i - \mu_i$.

This procedure is most easily understood with the aid of pictures. Suppose $\lambda = (4, 3, 2)$ and $\mu = (2, 1)$. Then

$\Delta_{\lambda/\mu} =$  $= \{(1, 3), (1, 4), (2, 2), (2, 3), (3, 1), (3, 2)\}.$

If $X = x_1 \wedge x_2 \otimes x_1 \wedge x_4 \otimes x_2 \wedge x_3$, then T_X "fills in" $\Delta_{\lambda/\mu}$ as follows:



Conversely, if T: $\Delta_{\lambda/\mu} \to \{x_1,..., x_n\}$ is in $\mathrm{Tab}_{\lambda/\mu}(\{x_1,..., x_n\})$, we obtain an element X_T of $\Lambda_{\lambda/\mu}$ defined by

$$X_T = x_{I_1} \otimes \cdots \otimes x_{I_q},$$

where $x_{I_i} = T(i, \mu_{i+1}) \wedge \cdots \wedge T(i, \lambda_i)$. Of course we do not necessarily have distinct elements in I_i, and even if distinct, they are not necessarily strictly increasing. Thus X_T is not necessarily a basis element of $\Lambda_{\lambda/\mu}F$. This leads us to make the following definitions.

DEFINITION II.2.2. A tableau $T \in \mathrm{Tab}_{\lambda/\mu}(S)$ is called *row-standard* if the rows of T are strictly increasing, i.e., if for all $i = 1,..., q$ we have $T(i, \mu_i + 1) < T(i, \mu_i + 2) < \cdots < T(i, \lambda_i)$. The tableau T is called *column-standard* if the columns of T are non-decreasing, i.e., if for all $j = 1,..., t$ $(t = \lambda_1)$ we have $T(i,j) \leqslant T(i+1,j)$ whenever (i,j) and $(i+1,j)$ are both in $\Delta_{\lambda/\mu}$. The tableau T is called *standard* if it is both row- and column-standard.

In the example above, the tableau is both row- and column-standar, hence standard.

We have seen that every basis element X of $\Lambda_{\lambda/\mu}F$ gives rise to a row-standard tableau T_X of $\mathrm{Tab}_{\lambda/\mu}(\{x_1,..., x_n\})$, and clearly every row-standard tableau T of $\mathrm{Tab}_{\lambda/\mu}\{x_1,..., x_n\}$ gives a basis element X_T of $\Lambda_{\lambda/\mu}F$. What we will show is that the set of elements $\{d_{\lambda/\mu}(X_T) \mid T$ a standard tableau in $\mathrm{Tab}_{\lambda/\mu}\{x_1,..., x_n\}\}$ is a basis for $L_{\lambda/\mu}F$. Since we have already seen that

$\{d_{\lambda/\mu}(X_T) \mid T$ a row-standard tableau in $\mathrm{Tab}_{\lambda/\mu}\{x_1,\ldots,x_n\}\}$ generates $L_{\lambda/\mu}F$, what we need first of all is a "straightening law" that shows that $d_{\lambda/\mu}(X_T)$, with T row-standard, is a linear combination of elements of the desired type. This will at least show that the images of the standard tableaux generate $L_{\lambda/\mu}F$. We will then finish the proof by showing that the images of the standard tableaux are linearly independent.

In order to obtain a straightening law, we must study the kernel of $d_{\lambda/\mu}$. Recall that the map $d_{\lambda/\mu}$ was defined explicitly as the composition

$$\Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q}F \to S_{a_{11}}F \otimes \cdots \otimes S_{a_{qt}}F$$
$$\to S_{a_{11}+\cdots+a_{q1}}F \otimes \cdots \otimes S_{a_{t1}+\cdots a_{qt}}F,$$

where

$$\begin{aligned} a_{ij} &= 0 \qquad \text{if } 1 \leqslant j \leqslant \mu_i \text{ or } \lambda_i + 1 \leqslant j \leqslant t \\ &= 1 \qquad \text{if } \mu_i + 1 \leqslant j \leqslant \lambda_i \end{aligned}$$

and $t = \lambda_1$. Fixing an i with $1 \leqslant i \leqslant q-1$, we observe:

LEMMA II.2.3. *The map $d_{\lambda/\mu}$ can be factored as follows:*

$$\begin{aligned} &\Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\lambda_i-\mu_i}F \otimes \Lambda^{\lambda_{i+1}-\mu_{i+1}}F \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q}F \\ &\quad \to^{\alpha} \Lambda^{\lambda_1-\mu_1}F \otimes \cdots \otimes S_{a_{i1}+a_{i+1\,1}}F \otimes \cdots \otimes S_{a_{it}+a_{i+1t}}F \otimes \Lambda^{\lambda_{i+2}-\mu_{i+2}}F \otimes \cdots \\ &\quad \otimes \Lambda^{\lambda_q-\mu_q}F \to^{\beta} S_{a_{11}}F \otimes \cdots \otimes S_{a_{i-1t}}F \otimes S_{a_{i1}+a_{i+1\,1}}F \otimes \cdots \\ &\quad \otimes S_{a_{it}+a_{i+1t}}F \otimes S_{a_{i+2\,1}}F \otimes \cdots \otimes S_{a_{qt}}F \to^{\gamma} S_{\tilde\lambda_1-\tilde\mu_1}F \otimes \cdots \otimes S_{\tilde\lambda_q-\tilde\mu_q}F, \end{aligned}$$

where $\alpha = 1 \otimes \cdots \otimes 1 \otimes d_{(\lambda_i,\lambda_{i+1})/(\mu_i,\mu_{i+1})} \otimes 1 \otimes \cdots \otimes 1$, *$\beta$ is the obvious diagonalization map on the remaining ΛF*'s, and *γ is multiplication in the tensor product of the symmetric algebras.*

As a result of Lemma II.2.3, we focus our attention on $d_{\lambda/\mu}$ when $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$. The skew diagram corresponding to λ/μ can also be described by the data $\{p_1, p_2, k\}$ where $p_1 = \lambda_1 - \mu_1$, $p_2 = \lambda_2 - \mu_2$ and $k = \lambda_2 - \mu_1$. In fact, if we let $\lambda' = (p_1 + p_2 - k, p_2)$ and $\mu' = (p_2 - k, 0)$, we see that the diagrams of λ/μ and λ'/μ' are the same (as are, therefore, the maps $d_{\lambda/\mu}$ and $d_{\lambda'/\mu'}$). The data $\{p_1, p_2, k\}$ describe the picture:

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whose top row is of length p_1, bottom row is of length p_2, and the shaded (or overlap) area is of length k.

DEFINITION II.2.4. Let $p_1, p_2, k \in \mathbb{N}$ with $p_i \geqslant k$. We define the map

$$\delta_k^{p_1,p_2}\colon \Lambda^{p_1}F \otimes \Lambda^{p_2}F \to \underbrace{S_2F \otimes \cdots \otimes S_2F}_{k} \otimes \Lambda^{p_1-k}F \otimes \Lambda^{p_2-k}F$$

as the composition of the maps

$$\Lambda^{p_1}F \otimes \Lambda^{p_2}F \xrightarrow{\Delta \otimes \Delta} \Lambda^kF \otimes \Lambda^{p_1-k}F \otimes \Lambda^kF \otimes \Lambda^{p_2-k}F \xrightarrow{\alpha} \Lambda^kF \otimes \Lambda^kF \otimes \Lambda^{p_1-k}F \otimes \Lambda^{p_2-k}F \xrightarrow{\beta} \underbrace{S_2F \otimes \cdots \otimes S_2F}_{k} \otimes \Lambda^{p_1-k}F \otimes \Lambda^{p_2-k}F,$$

where Δ is the indicated diagonal map, α simply permutes terms, and β is $d_{(k,k)} \otimes 1 \otimes 1$.

As an easy consequence of the definitions, and the (co-)associativity and (co-)commutativity of the Hopf algebras involved, we have:

LEMMA II.2.5. $\delta_{k+l}^{p_1,p_2} = (1 \otimes \delta_l^{p_1-k,p_2-k}) \circ \delta_k^{p_1,p_2}$.

LEMMA II.2.6. *Let a be the matrix*

$$\begin{matrix} 1\cdots 1 & 1\cdots 1 & 0\cdots 0 \\ \underbrace{1\cdots 1}_{k} & \underbrace{0\cdots 0}_{p_1-k} & \underbrace{1\cdots 1}_{p_2-k} \end{matrix}.$$

Then the diagram

$$\begin{array}{ccc} & \Lambda^{p_1}F \otimes \Lambda^{p_2}F & \\ {\scriptstyle \delta_k^{p_1,p_2}} \swarrow & & \searrow {\scriptstyle d_\alpha} \\ S_2F \otimes \cdots \otimes S_2F \otimes \Lambda^{p_1-k}F \otimes \Lambda^{p_2-k}F & \xrightarrow{1 \otimes \Delta \otimes \Delta} & S_{\tilde{\alpha}}F \end{array}$$

is commutative. Moreover, since the bottom map is an injection, it follows that $\operatorname{Im} \delta_k^{p_1,p_2} \cong \operatorname{Im} d_\alpha$. *Finally, if we set* $\lambda = (p_1 + p_2 - k, p_2)$, *and* $\mu = (q_2 - k, 0)$, *it follows from Lemma II.1.5 that* $\operatorname{Im} \delta_k^{p_1,p_2} \approx L_{\lambda/\mu}$.

LEMMA II.2.7. *Let p_1, p_2, k be positive integers, with $p_i \geqslant k$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \Lambda^{p_1+p_2-k}F \otimes \Lambda^kF & \xrightarrow{\square_k} & \Lambda^{p_1}F \otimes \Lambda^{p_2}F \\ \downarrow {\scriptstyle \delta_1^{p_1+p_2-k,k}} & & \downarrow {\scriptstyle \delta_1^{p_1,p_2}} \\ S_2F \otimes \Lambda^{p_1+p_2-k-1}F \otimes \Lambda^{k-1}F & \xrightarrow{1 \otimes \square_{k-1}} & S_2F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-1}F, \end{array}$$

where $\square_k$ *is the composite map:*

$$\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\Delta \otimes 1} \Lambda^{p_1}F \otimes \Lambda^{p_2-k}F \otimes \Lambda^k F \xrightarrow{1 \otimes m} \Lambda^{p_1}F \otimes \Lambda^{p_2}F.$$

The map Δ *is the appropriate diagonal, and* m *is multiplication.*

Proof. Let $u \otimes v \in \Lambda^{p_1+p_2-k}F \otimes \Lambda^k F$. Then $\delta_1^{p_1+p_2-k,k}(u \otimes v) = \sum u_i' v_j' \otimes u_i \otimes v_j$, where $\Delta(u) = \sum u_i \otimes u_i' \in \Lambda^{p_1+p_2-k-1}F \otimes F$ and $\Delta(v) = \sum v_j' \otimes v_j \in F \otimes \Lambda^{k-1}F$. Thus $(1 \otimes \square_{k-1})\, \delta_1^{p_1+p_2-k,k}\,(u \otimes v) = \sum u_i' v_j' \otimes u_{is}'' \otimes u_{is}''' \wedge v_j$, where

$$\Delta(u_i) = \sum u_{is}'' \otimes u_{is}''' \in \Lambda^{p_1-1}F \otimes \Lambda^{p_2-k}F.$$

Because of coassociativity of the diagonal, we may therefore write $(1 \otimes \square_{k-1})\, \delta_1^{p_1+p_2-k,k}(u \otimes v) = \sum u_t' v_j' \otimes u_t \otimes u_t'' \wedge v_j$, where $\Delta^2(u) = \sum u_t \otimes u_t' \otimes u_t'' \in \Lambda^{p_1-1}F \otimes F \otimes \Lambda^{p_2-k}F$.

Computing the other route around our diagram, we have $\square_k(u \otimes v) = \sum u_\alpha \otimes u_\alpha' \wedge v$, where $\Delta(u) = \sum u_\alpha \otimes u_\alpha' \in \Lambda^{p_1}F \otimes \Lambda^{p_2-k}F$. Now $\delta_1^{p_1,p_2}(\sum u_\alpha \otimes u_\alpha' \wedge v) = \sum u_t' v_j' \otimes u_t \otimes u_t'' \wedge v_j + \sum u_t' u_{tl}''' \otimes u_t \otimes u_{tl}'' \wedge v$, where $\Delta(u_t'') = \sum u_{tl}''' \otimes u_{tl}'' \in F \otimes \Lambda^{p_2-k-1}$. But using coassociativity and cocommutativity of Δ again, we see that the term $\sum u_t' u_{tl}''' \otimes u_t \otimes u_{tl}'' \wedge v = \sum u_\beta' u_\beta'' \otimes u_\beta \otimes u_\beta''' \wedge v$, where

$$\Delta^3(u) = \sum u_\beta \otimes u_\beta' \otimes u_\beta'' \otimes u_\beta''' \in \Lambda^{p_1-1}F \otimes F \otimes F \otimes \Lambda^{p_2-k-1}F.$$

However, the element $\sum u_\beta' u_\beta'' \otimes u_\beta \otimes u_\beta''' \in S_2 F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-k-1}F$ is precisely the image of u under the composite map

$$\Lambda^{p_1+p_2-k}F \xrightarrow{\Delta} \Lambda^{p_1}F \otimes \Lambda^{p_2-k}F \xrightarrow{\delta_1^{p_1,p_2-k}} S_2 F \otimes \Lambda^{p_1-1}F \otimes \Lambda^{p_2-k-1}F,$$

and we know, from [3], that this composition is zero. Hence, $\sum u_\beta' u_\beta'' \otimes u_\beta \otimes u_\beta''' \wedge v = 0$ and our diagram is commutative. This completes the proof of II.2.7.

PROPOSITION II.2.8. *Let* $p_1, p_2, k \in \mathbb{N}$ *with* $p_i \geqslant k+1$. *Then the composition*

$$\Lambda^{p_1+p_2-k}F \otimes \Lambda^k F \xrightarrow{\square_k} \Lambda^{p_1}F \otimes \Lambda^{p_2}F \xrightarrow{\delta_{k+1}^{p_1,p_2}} \underbrace{S_2 F \otimes \cdots \otimes S_2 F}_{k+1}$$

$$\otimes \Lambda^{p_1-k-1}F \otimes \Lambda^{p_2-k-1}F$$

is zero.

Proof. We proceed by induction on k. When $k=0$, we want to show that the composition

$$\Lambda^{p_1+p_2}F \xrightarrow{\Delta} \Lambda^{p_1}F \otimes \Lambda^{p_2}F \xrightarrow{\Delta\otimes\Delta} \Lambda^{p_1-1}F \otimes F \otimes F \otimes \Lambda^{p_2-1}F$$
$$\to \Lambda^{p_1-1}F \otimes S_2F \otimes \Lambda^{p_2-1}F$$

is zero. As in the last step of the proof of II.2.7, we may invoke [3]. However, an alternative way of seeing this (and this applies, obviously, to II.2.7 also) is as follows. By coassociativity of the diagonal, the composition $(\Delta \otimes \Delta) \circ \Delta$ above equals the composition

$$\Lambda^{p_1+p_2}F \xrightarrow{\Delta^2} \Lambda^{p_1-1}F \otimes \Lambda^2F \otimes \Lambda^{p_2-1}F \xrightarrow{1\otimes\Delta\otimes 1} \Lambda^{p_1-1}F \otimes F \otimes F \otimes \Lambda^{p_2-1}F.$$

The result now follows from the fact that

$$\Lambda^2F \to F \otimes F \to S_2F$$

is zero.

Now let $k>0$. Then

$$\begin{aligned}\delta_{k+1}^{p_1,p_2}\,\square_k &= (1\otimes \delta_k^{p_1-1,p_2-1}) \circ \delta_1^{p_1,p_2} \circ \square_k\\ &= (1\otimes \delta_k^{p_1-1,p_2-1}) \circ (1 \otimes \square_{k-1})\, \delta_1^{p_1+p_2-k,k},\end{aligned}$$

and this last composition is zero by induction.

We now require one more lemma before we can obtain our straightening law.

LEMMA II.2.9. *Let p_1, p_2, k, u, l be nonnegative integers such that $p_i \geqslant k$ for $i=1,2$, $0 \leqslant u \leqslant l \leqslant k-1$. Denote by $\bar{\square}_u$ the composite map*

$$\Lambda^uF \otimes \Lambda^{p_1+p_2-l}F \otimes \Lambda^{l-u}F \xrightarrow{1\otimes\Delta\otimes 1} \Lambda^uF \otimes \Lambda^{p_1-u}F$$
$$\otimes \Lambda^{p_2+u-l}F \otimes \Lambda^{l-u}F \xrightarrow{m\otimes m} \Lambda^{p_1}F \otimes \Lambda^{p_2}F.$$

Then $\mathrm{Im}(\bar{\square}_u)$ *is contained in the image of the map*

$$\square_k^{p_1,p_2} = \sum_{v=0}^{k-1} \square_v : \sum_{v=0}^{k-1} \Lambda^{p_1+p_2-v}F \otimes \Lambda^vF \to \Lambda^{p_1}F \otimes \Lambda^{p_2}F.$$

Proof. When $u=0$, $\bar{\square}_0' = \square_l$, so there is nothing to prove. Assume that

$u > 0$, and use induction. Let $x \otimes y \otimes z \in \Lambda^u F \otimes \Lambda^{p_1+p_2-l} F \otimes \Lambda^{l-u} F$. Then $\overline{\square}_u(x \otimes u \otimes z) = \sum x \wedge y_i' \otimes y_i \wedge z$, where

$$\Delta(y) = \sum y_i' \otimes y_i \in \Lambda^{p_1-u} F \otimes \Lambda^{p_2+u-l} F.$$

Since $x \wedge y \otimes z \in \Lambda^{p_1+p_2-(l-u)} F \otimes \Lambda^{l-u} F$, we may consider $\square_{l-u}(x \wedge y \otimes z)$ and this gives us $\sum x \wedge y_i \otimes y_i' \wedge z + \sum x_{s\alpha} \wedge y_{t\alpha} \otimes x_{s\alpha}' \wedge y_{t\alpha}' \wedge z$, where

$$\Delta(x) = \sum_{\alpha=1}^{u} \sum x_{s\alpha} \otimes x_{s\alpha}' \in \sum_{\alpha=1}^{u} \Lambda^{u-\alpha} F \otimes \Lambda^{\alpha} F,$$

and

$$\Delta(y) = \sum_{\alpha=1}^{u} \sum y_{t\alpha} \otimes y_{t\alpha}' \in \sum_{\alpha=1}^{u} \Lambda^{p_1-(u-\alpha)} F \otimes \Lambda^{p_2-(l-u+\alpha)} F.$$

We therefore see that

$$\square_{l-u}(x \wedge y \otimes z) - \overline{\square}_u(x \otimes y \otimes z) = \sum x_{s\alpha} \wedge y_{t\alpha} \otimes x_{s\alpha}' \wedge y_{t\alpha}' \wedge z.$$

For each α, consider the element

$$w_\alpha = \sum x_{s\alpha} \otimes y \otimes x_{s\alpha}' \wedge z \in \Lambda^{u-\alpha} F \otimes \Lambda^{p_1+p_2-l} F \otimes \Lambda^{l-(u-\alpha)}.$$

Clearly, $\sum_{\alpha=1}^{u} \pm \overline{\square}_{u-\alpha}(w_\alpha) = \square_{l-u}(x \wedge y \otimes z) - \overline{\square}_u(x \otimes y \otimes z)$. However, our induction hypothesis on u tells us that $\overline{\square}_{u-\alpha}(w_\alpha) \subset \mathrm{Im}(\square_k^{p_1,p_2})$ and this completes the proof.

Definition II.2.10. If $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ are partitions with $\mu \subset \lambda$, we will denote by $\square_{\lambda/\mu}$ the map $\square_k^{p_1,p_2}$, where $p_i = \lambda_i - \mu_i$ and $k = \lambda_2 - \mu_1$. If $\lambda = (\lambda_1, \ldots, \lambda_q)$ and $\mu = (\mu_1, \ldots, \mu_q)$ are partitions with $\mu \subset \lambda$, then for each $i = 1, \ldots, q-1$ we have the partitions $\lambda^i = (\lambda_i, \lambda_{i+1})$ and $\mu^i = (\mu_i, \mu_{i+1})$ and the maps $1_1 \otimes \cdots \otimes 1_{i-1} \otimes \square_{\lambda^i/\mu^i} \otimes 1_{i+2} \otimes \cdots \otimes 1_q$ into $\Lambda^{\lambda_1-\mu_1} F \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q} F$, where the maps 1_j are the identities on $\Lambda^{\lambda_j-\mu_j} F$ for $j \neq i, i+1$. Define $\square_{\lambda/\mu}$ to be the sum of all of these maps, i.e.,

$$\square_{\lambda/\mu} = \sum_{i=1}^{q-1} 1_1 \otimes \cdots \otimes 1_{i-1} \otimes \square_{\lambda^i/\mu^i} \otimes 1_{i+2} \otimes \cdots \otimes 1_q.$$

Theorem II.2.11. *The image of* $\square_{\lambda/\mu}$ *is contained in the kernel of* $d_{\lambda/\mu}$.

Proof. Clear.

DEFINITION II.2.12. For partitions $\mu \subset \lambda$, define $\bar{L}_{\lambda/\mu}(F)$ to be the cokernel of the map $\square_{\lambda/\mu}$.

If we let $\bar{d}_{\lambda/\mu}: \Lambda_{\lambda/\mu}F \to \bar{L}_{\lambda/\mu}(F)$ be the canonical surjection, then clearly there is a unique surjection $\theta_{\lambda/\mu}: \bar{L}_{\lambda/\mu}F \to L_{\lambda/\mu}F$ such that $\theta_{\lambda/\mu}\bar{d}_{\lambda/\mu} = d_{\lambda/\mu}$.

Before starting the next theorem, we need to make one further definition and prove two lemmas.

DEFINITION II.2.13. Let $T \in \mathrm{Tab}_{\lambda/\mu}\{x_1,..., x_n\}$, and let p, q be positive integers. Define $T_{p,q}$ to be the number of times the elements $x_1,..., x_q$ appear as entries in the first p rows of T. More precisely, $T_{p,q} = \#\{(i,j) \in T_{\lambda/\mu} \mid i \leqslant p$ and $T(i,j) \in \{x_1,..., x_q\}\}$. If S is another tableau, we say $S \leqslant T$ if $S_{p,q} \geqslant T_{p,q}$ for every p, q. We say $S < T$ if $S \leqslant T$ and $S_{p,q} > T_{p,q}$ for at least one pair p, q. We define the content of a tableau T to be the sequence in $\mathbb{N}^{\infty}$ whose ith term is the number of times the element x_i appears as an entry in T.

This puts a pseudo-order on the set of tableaux; for it is clearly reflexive and transitive, but equally clearly we can have $S \leqslant T$ and $T \leqslant S$ without $S = T$. Notice that if S is obtained from T by row-standardization, then $S_{p,q} = T_{p,q}$ for all p, q. Observe, too, that if we restrict this pseudo-order to the subset of row-standard tableaux, then on this subset the order is consistent with the usual lexicographic (total) order induced by the correspondence between row-standard tableaux and basis elements of $\Lambda_{\lambda/\mu}(F)$.

The canonical tableau in $\mathrm{Tab}_{\lambda}\{x_1,..., x_n\}$ is the tableau C_{λ} defined by $C_{\lambda}(i,j) = x_j$. The content of C_{λ} is clearly λ and if T is any other standard tableau in $\mathrm{Tab}_{\lambda}\{x_1,..., x_n\}$, then $C_{\lambda} < T$.

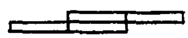
LEMMA II.2.14. *Let $T \in \mathrm{Tab}_{\lambda/\mu}\{x_1,..., x_n\}$, and let S be the tableau, also of shape λ/μ, formed by exchanging certain entries from the kth row of T, say $T(k, l_1),..., T(k, l_{\alpha})$, with certain entries of the $(k+1)$st row of T, say $T(k+1, m_1),..., T(k+1, m_{\alpha})$, where $T(k+1, m_{\nu}) < T(k, l_{\nu})$ for $\nu = 1,..., \alpha$. More precisely, for $(i,j) \notin \{(k, l_1),..., (k, l_{\alpha}), (k+1, m_1),..., (k+1, m_{\alpha})\}$, $S(i,j) = T(i,j)$, but $S(k, l_{\nu}) = T(k+1, m_{\nu})$ and $S(k+1, m_{\nu}) = T(k, l_{\nu})$ for $\nu = 1,..., \alpha$. Then $S < T$.*

Proof. It is clearly enough to consider the case $\alpha = 1$. Thus we have tableaux T and S with $T(i,j) = S(i,j)$ for all (i,j) other than (k, l) and $(k+1, m)$. In these two places we have $S(k, l) = T(k+1, m)$ and $S(k+1, m) = T(k, l)$. If we set $x_b = T(k, l)$ and $x_a = T(k+1, m)$, we assume also that $a < b$, where a and b are integers between 1 and n. Clearly, $S_{p,q} = T_{p,q}$ for $p \neq k$. But it is also easy to check that $S_{k,q} = T_{k,q}$ for $1 \leqslant q < a$ and $b \leqslant q \leqslant n$; $S_{k,q} = T_{k,q} + 1$ for $a \leqslant q < b$. Thus $S < T$.

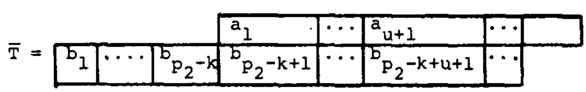
LEMMA II.2.15. *If T is a row-standard tableau which is not standard,* *then there exist row-standard tableaux T_l with $T_l < T$ such that $X_T - \sum \pm X_{T_l} \in \operatorname{Im} \square_{\lambda/\mu}$. In particular, $d_{\lambda/\mu}(X_T) = \sum \pm d_{\lambda/\mu}(X_{T_l})$.*

Proof. If $T = (T^1,..., T^q)$ is row-standard but not standard, then there are two adjacent rows of T in which column-standardness is violated, say rows T^j and T^{j+1}. We will show that by modifying just these two rows, we get row-standard tableaux $T_l < T$ with $X_T - \sum \pm X_{T_l} \in \operatorname{Im}(\square_{\lambda/\mu})$. In fact, what we do is consider the partitions $\lambda^j = (\lambda_j, \lambda_{j+1})$ and $\mu^j = (\mu_j, \mu_{j+1})$. The rows (T^j, T^{j+1}) then give us a row-standard tableau $\bar{T}$ in $\operatorname{Tab}_{\lambda^j/\mu^j}$. We will first show that we can find $\bar{T}_l \in \operatorname{Tab}_{\lambda^j/\mu^j}$ such that $X_{\bar{T}} - \sum X_{\bar{T}_l} \in \operatorname{Im} \square_{\lambda^j/\mu^j}$.

Suppose that the diagram of λ^j/μ^j is the following:

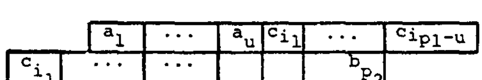
 .

We will set $\lambda_j - \mu_j = p_1, \lambda_{j+1} - \mu_{j+1} = p_2$ and $\lambda_{j+1} - \mu_j = k$, as usual. For convenience, we will call the basis elements in the top row $a_1,..., a_{p_1}$ and those in the bottom row $b_1,..., b_{p_2}$. Since we are assuming column-standardness to be violated, let us take a_{u+1} to be the first position in which the violation takes place. We then have the following picture:



Our assumptions are that the rows are strictly increasing, and that $a_i \leqslant b_{p_2-k+i}$ for $i = 1,..., u$, but that $a_{u+1} > b_{p_2-k+u+1}$. Consider the element $w = a_1 \wedge \cdots a_u \otimes b_1 \wedge \cdots \wedge b_{p_2-k+u+1} \wedge a_{u+1} \wedge \cdots \wedge a_{p_1} \otimes b_{p_2-k+u+2} \wedge \cdots \wedge b_{p_2}$ in $\Lambda^u F \otimes \Lambda^{p_1+p_2-k+1} F \otimes \Lambda^{l-1-u} F$. We are now in the situation anticipated by Lemma II.2.9 with $l = k - 1$, and we know that $\bar{\square}_u(w) \in \operatorname{Im}(\square_k^{p_1,p_2}) = \operatorname{Im}(\square_{\lambda^j/\mu^j})$.

Now $\bar{\square}_u(w) = \sum \pm a_1 \wedge \cdots \wedge a_u \wedge c_I \otimes c_{\bar{I}} \wedge b_{p_2-k+u+2} \wedge \cdots \wedge b_{p_2}$, where c_I is the product of $p_1 - u$ terms of $b_1 \wedge \cdots \wedge b_{p_2-k+u+1} \wedge a_{u+1} \wedge \cdots \wedge a_{p_1}$ and $c_{\bar{I}}$ is the product of the complementary terms. Notice that only one term, c_I, can consist only of the product $a_{u+1} \wedge \cdots \wedge a_{p_1}$. All the other c_I must have at least one b_i in it with $1 \leqslant i \leqslant p_2 - k + u + 1$. If we denote by $\bar{T}_I$ the row-standardization of the tableau whose first row has the entries $a_1,..., a_u, c_{i_1},..., c_{i_{p_1-u}}$, and whose second row has the entries $c_{\bar{i}_1},..., b_{p_2}$, that is,

 ,

then precisely one of these tableaux is our original tableau $\bar{T}$ (say, $\bar{T}_{I_0}$), while all the others either have repeats in a row, or, when row-standardized, are obtained from the original tableau by exchanging some of the entries $a_{u+1},\dots, a_{p_1}$ by some of $b_1,\dots, b_{p_2-k+u+1}$. Thus, by Lemma II.2.14, each of the corresponding tableaux is smaller than $\bar{T}$. Thus

$$X_{\bar{T}_{I_0}} - \sum_{I \neq I_0} X_{\bar{T}_I} \in \operatorname{Im} \square_{\lambda^j/\mu^j}$$

and we have proven the lemma for the two-rowed diagram.

The proof for the general case is now straightforward. We simply modify the rows T^j and T^{j+1} of T by inserting the row-standard tableaux $\bar{T}_I$ in their stead. The tableaux T_I thus obtained differ from T only by an exchange in adjacent rows j and $j+1$ of the sort described in Lemma II.2.14. Thus each such $T_I < T$ and clearly $X_T - \sum \pm X_{T_I} \in \operatorname{Im} \square_{\lambda/\mu}$. This completes the proof of Lemma II.2.15

THEOREM II.2.16. *Let $\lambda = (\lambda_1,\dots, \lambda_q)$, $\mu = (\mu_1,\dots, \mu_q)$ be partitions with $\mu \subset \lambda$, and let F be a free module with ordered basis $\{x_1,\dots, x_n\}$. Then $\{d_{\lambda/\mu}(X_T) \mid T$ is a standard tableau in $\operatorname{Tab}_{\lambda/\mu}\{x_1,\dots, x_n\}\}$ is a free basis for $L_{\lambda/\mu}(F)$, and the map $\theta_{\lambda/\mu}\colon \bar{L}_{\lambda/\mu}(F) \to L_{\lambda/\mu}(F)$ is an isomorphism. Hence $L_{\lambda/\mu}(F)$ is universally free.*

Proof. We claim that Lemma II.2.15 implies that the set $\{d_{\lambda/\mu}(X_T)/T$ standard$\}$ generates $L_{\lambda/\mu}(F)$ and $\{\bar{d}_{\lambda/\mu}(X_T)/T$ standard$\}$ generates $\bar{L}_{\lambda/\mu}(F)$. For, if we start with a non-standard, row-standard tableau T, we may write

$$d_{\lambda/\mu}(X_T) = \sum \pm d_{\lambda/\mu}(X_{T_i})$$

with $T_i < T$ (and similarly for $\bar{d}_{\lambda/\mu}$). If T_i is not standard, we may repeat this procedure. However, since $\operatorname{Tab}_{\lambda/\mu}\{x_1,\dots, x_n\}$ is a finite set, we must ultimately arrive at standard tableaux, and this gives us our first assertion.

Having proved that $\{d_{\lambda/\mu}(X_T)/T$ standard$\}$ generates $L_{\lambda/\mu}(F)$ [and $\bar{L}_{\lambda/\mu}(F)$], we must now show that these are linearly independent.

First, we make a trivial observation. If T is in $\operatorname{Tab}_{\lambda/\mu}\{x_1,\dots, x_n\}$, we can associate to it an element Z_T in $S_{\tilde{\lambda}/\tilde{\mu}}$ by taking the product of the elements in the jth column of T in $S_{\tilde{\lambda}_j-\tilde{\mu}_j}$, similar to the way we associated the element X_T in $\Lambda_{\lambda/\mu}$ to the tableau T. Clearly, the column-standard tableaux correspond precisely to the basis elements of $S_{\tilde{\lambda}/\tilde{\mu}}$. If T is a tableau, we can, by rearranging the terms in each column to be non-decreasing (that is, by column-standardization of T), obtain a column-standard tableau T' with $Z_T = Z_{T'}$.

Next, observe that if $a = a_1 \wedge \cdots \wedge a_p \in \Lambda^p F$, then the p-fold diagonalization of a in

$$\underbrace{F \otimes \cdots \otimes F}_{p}$$

is $\sum_\sigma (-1)^\sigma a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(p)}$, where σ runs through all permutations of $\{1,\dots,p\}$. Thus, if T is in $\mathrm{Tab}_{\lambda/\mu}\{x_1,\dots,x_n\}$, say T has $a_{i1},\dots,a_{i\lambda_i-\mu_i}$ in its ith row, then $d_{\lambda/\mu}(X_T) = \sum_{\sigma=(\sigma_1,\dots,\sigma_q)} \pm Z_{T_\sigma}$, where $T_\sigma \in \mathrm{Tab}_{\lambda/\mu}\{x_1,\dots,x_n\}$ and the ith row of T_σ is $a_{i\sigma_i(1)},\dots,a_{i\sigma_i(\lambda_i-\mu_i)}$ with σ_i a permutation of $\{1,\dots,\lambda_i-\mu_i\}$.

Now let $\Lambda'_{\lambda/\mu}$ be the submodule of $\Lambda_{\lambda/\mu}$ generated by $\{X_T/T$ standard in $\mathrm{Tab}_{\lambda/\mu}\{x_1,\dots,x_n\}\}$, and let $S'_{\tilde{\lambda}/\tilde{\mu}}$ be the submodule generated by $\{Z_T/T$ standard in $\mathrm{Tab}_{\lambda/\mu}\{x_1,\dots,x_n\}\}$. These are free modules, and we have the injection $i\colon \Lambda'_{\lambda/\mu} \to \Lambda_{\lambda/\mu}$ and the projection $\pi\colon S_{\tilde{\lambda}/\tilde{\mu}} \to S'_{\tilde{\lambda}/\tilde{\mu}}$. The map $\pi d_{\lambda/\mu} i$ is a map of free modules

$$\pi \circ d_{\lambda/\mu} \circ i\colon \Lambda'_{\lambda/\mu} \to S'_{\tilde{\lambda}/\tilde{\mu}},$$

and both modules have bases indexed by the same set, namely, the standard tableaux. The set of standard tableaux is totally ordered lexicographically. We will show that with respect to these bases in their lexicographic order, the matrix of the map $\pi \circ d_{\lambda/\mu} \circ i$ is triangular with ones on the diagonal. To this end, suppose that T is standard. Then $d_{\lambda/\mu} \circ i(X_T) = \sum_{\sigma=(\sigma_1,\dots,\sigma_q)} \pm Z_{T_\sigma}$ as described above. If we denote by T'_σ the column-standardization of T_σ, then clearly

$$\pi \circ d_{\lambda/\mu} \circ i(X_T) = \sum \pm Z_{T'_\sigma}, \tag{*}$$

where σ now ranges over those permutations such that T'_σ is standard. We claim that for all $\sigma \neq 1$ such that T'_σ is standard, $T'_\sigma < T$ in the pseudo-order we have been using. We see this as follows. First, if $T_\sigma \neq T'_\sigma$, then by Lemma II.2.14 we must have $T'_\sigma < T_\sigma$ and hence, $T'_\sigma < T$ (since T_σ and T have the "same order"). Next, if T'_σ is row-standard and $\sigma \neq 1$, then $T_\sigma \neq T'_\sigma$ since T_σ itself cannot be row-standard if $\sigma \neq 1$. Thus, by the preceding step, $T'_\sigma < T$. The remark following the definition of our pseudo-order tells us that this pseudo-order is consistent with the lexicographic order on standard tableaux. Hence, since Z_T clearly occurs in the sum (*) with coefficient 1, and all the other summands in (*) are indexed by T'_σ with $T'_\sigma < T$, we obtain the triangularity of the matrix of $\pi \circ d_{\lambda/\mu} \circ i$ with respect to our given bases.

Knowing that the matrix of $\pi \circ d_{\lambda/\mu} \circ i$ is triangular with ones on the diagonal, we conclude that this map is an isomorphism. Since $L_{\lambda/\mu}(F)$ is the image of $d_{\lambda/\mu} \circ i$, it follows that $L_{\lambda/\mu}(F)$ is isomorphic to $\Lambda'_{\lambda/\mu}$, and has the basis $\{d_{\lambda/\mu}(X_T)/T$ standard$\}$. It now also follows trivially that the surjective map $\theta_{\lambda/\mu}$ is an isomorphism, so that $\bar{L}_{\lambda/\mu}(F)$ is free too. But $\bar{L}_{\lambda/\mu}(F)$ is the cokernel of a map between universally free modules so that, being free, it is universally free as is, therefore, $L_{\lambda/\mu}(F)$. This concludes the proof of the theorem.

II.3. *The Universal Freeness of Coschur Functors*

Until now, we have been concentrating on Schur functors, and have said nothing about coSchur functors. Recall that the coSchur functor $K_{\lambda/\mu}F$ is defined (Definition II.1.4) as the image of the map $d'_{\lambda/\mu}: D_{\lambda/\mu}F \to \Lambda_{\tilde{\lambda}/\tilde{\mu}}F$. For the most part, the proof of universal freeness and the description of the standard basis of $K_{\lambda/\mu}F$ in terms of a given basis $\{x_1,..., x_n\}$ of F proceed formally as in the case of $L_{\lambda/\mu}F$, but a few modifications and amplifications are required, and we will describe them here.

Retaining the definition of $\mathrm{Tab}_{\lambda/\mu}(S)$ (Definition II.2.1), we modify Definition II.2.2 as follows:

DEFINITION II.3.2. A tableau $T \in \mathrm{Tab}_{\lambda/\mu}(S)$ is called *co-row-standard* if the rows of T are non-decreasing, and *co-column-standard* if the columns of T are strictly increasing. T is called *co-standard* if it is co-row- and co-column-standard.

If $I = \{1 \leqslant i_1 \leqslant \cdots \leqslant i_q \leqslant n\}$ is a non-decreasing sequence of integers, we may group these integers into distinct clumps: $i_1 = i_2 = \cdots = i_{t_1} < i_{t_1+1} \cdots = i_{t_2} < \cdots < i_{t_l+1} = \cdots = i_q$. If $\{x_1,..., x_n\}$ is a basis for F, we obtain a basis element of D_qF, x_I, by setting $x_I = x_{i_1}^{(t_1)} \cdots x_{i_{t_1+1}}^{(t_2-t_1)} \cdots x_{i_{t_l}}^{(q-t_l)}$. With this assignment, it is clear that co-row-standard tableaux correspond to basis elements of $D_{\lambda/\mu}F$ and co-column-standard tableaux correspond (in the obvious way) to basis elements of $\Lambda_{\tilde{\lambda}/\tilde{\mu}}F$. What we will prove (in outline only) is that the co-standard tableaux are a basis for $K_{\lambda/\mu}F$.

Lemma II.2.3 has a counterpart with $d_{\lambda/\mu}$ replaced by $d'_{\lambda/\mu}$. Definition II.2.4 may be formally replaced by:

DEFINITION II.3.4. $(\delta_k^{p_1,p_2})'$: $D_{p_1}F \otimes D_{p_2}F \to \underbrace{\Lambda^2F \otimes \cdots \otimes \Lambda^2F}_{k} \otimes D_{p_1-k}F \otimes D_{p_2-k}F$.

Lemmas II.2.5 through II.2.7 also have formal co-statements, as does Proposition II.2.8, where we define $\square'_k$ as the composite map

$$D_{p_1+p_2-k}F \otimes \Lambda^kF \xrightarrow{\Delta \otimes 1} D_{p_1}F \otimes D_{p_2-k}F \otimes D_kF \xrightarrow{1 \otimes m} D_{p_1}F \otimes D_{p_2}F.$$

(The reference [3] in the proof of II.2.7 has its dual counterpart in $[\bar{3}]$.)

In the proof of II.2.9, we define maps $\square'_u$ as counterparts of the maps $\bar{\square}_u$, and for II.2.10 we define the map $\square'_{\lambda/\mu}$ in the obvious way. This yields:

Theorem II.3.11. *The image of $\square'_{\lambda/\mu}$ is contained in the kernel of $d'_{\lambda/\mu}$.*

Definition II.3.12. For partitions $\mu \subset \lambda$, define $\bar{K}_{\lambda/\mu}(F)$ to be the cokernel of the map $\square'_{\lambda/\mu}$, and let $\bar{d}'_{\lambda/\mu}\colon D_{\lambda/\mu}F \to \bar{K}_{\lambda/\mu}F$ be the canonical surjection.

We then obtain a unique surjection $\theta'_{\lambda/\mu}\colon \bar{K}_{\lambda/\mu}F \to K_{\lambda/\mu}F$ such that $\theta'_{\lambda/\mu}\bar{d}'_{\lambda/\mu} = d'_{\lambda/\mu}$.

Theorem II.2.16 now has the dual reading:

Theorem II.3.16. *Let $\lambda = (\lambda_1,..., \lambda_q)$, $\mu = (\mu_1,..., \mu_q)$ be partitions with $\mu \subseteq \lambda$, and let F be a free module with ordered basis $\{x_1,..., x_n\}$. Then $\{d'_{\lambda/\mu}(X_T)/T$ is a co-standard tableau in $\mathrm{Tab}_{\lambda/\mu}\{x_1,..., x_n\}\}$ is a free basis for $K_{\lambda/\mu}(F)$, and the map $\theta'_{\lambda/\mu}\colon \bar{K}_{\lambda/\mu}(F) \to K_{\lambda/\mu}(F)$ is an isomorphism. Hence, $K_{\lambda/\mu}(F)$ is universally free.*

(In this statement, X_T is the basis element of $D_{\lambda/\mu}F$ that corresponds to the co-standard tableau T under the obvious extension of the correspondence described in the paragraph immediately following Definition II.3.2.)

The proof is, formally, almost exactly the same as that of II.2.16. The only slight modification is in the proof of II.2.15. The reduction of the problem to the two-rowed tableau

$\bar{T}$ =

			a_1	...	a_{u+1}	...	a_{p_1}
b_1	...	b_{p_2-k}	b_{p_2-k+1}	...	$b_{p_2-k+u+1}$	...	

proceeds as in the proof of II.2.15, and we have the co-row-standard tableau $\bar{T}$ above, with the first violation of co-column-standardness occurring between the elements a_{u+1} and $b_{p_2-k+u+1}$. Thus, the rows are non-decreasing, and $a_i < b_{p_2-k+i}$ for $i = 1,..., u$, but $a_{u+1} \geqslant b_{p_2-k+u+1}$. Since $a_u < b_{p_2-k+u} \leqslant b_{p_2-k+u+1} \leqslant a_{u+1}$, we have $a_1 \leqslant \cdots \leqslant a_u < a_{u+1} \leqslant a_{u+2} \cdots$. Now let h be the largest integer such that $b_{p_2-k+u+1} = b_h$. Then $a_1 \leqslant \cdots \leqslant a_u$, $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{p_2-k+u+1} = \cdots = b_h \leqslant a_{u+1} \leqslant \cdots \leqslant a_{p_1}$, and $b_{h+1} \leqslant \cdots \leqslant b_{p_2}$ are non-decreasing sequences of basis elements of F and therefore define basis elements $\alpha \in D_u F$, $\beta \in D_{h+p_1-u}F$ and $\gamma \in D_{p_2-h}F$, respectively. (Note that it is precisely here where our proof differs from that of II.2.15, as we had to go all the way up to b_h to form the element β, because of possible repeats in the rows.) The element $\alpha \otimes \beta \otimes \gamma$ is in $D_u F \otimes D_{h+p_1-u}F \otimes D_{p_2-h}F = D_u F \otimes D_{p_1+p_2-(p_2-h+u)}F \otimes D_{(p_2-h+u)-u}F$, and we are in the situation anticipated by Lemma II.2.9 (with $l = p_2 - h + u$). Hence, we know that $\bar{\square}'_u(\alpha \otimes \beta \otimes \gamma) \in \mathrm{Im}(\square'_{\lambda^j/\mu^j})$, where λ^j/μ^j is the two-rowed skew-partition to which Lemma II.2.15 refers.

To compute $\bar{\square}'_u(\alpha \otimes \beta \otimes \gamma)$, we first diagonalize β in $D_{p_1-u}(F) \otimes D_h(F)$, i.e., $\Delta(\beta)=\sum \beta_i \otimes \beta_i'$ with $\beta_i \in D_{p_1-u}(F)$, $\beta_i' \in D_h(F)$. Then $\bar{\square}_u'(\alpha \otimes \beta \otimes \gamma)=\sum \alpha\beta_i \otimes \beta_i'\gamma$, where multiplication occurs in $D(F)$ (and, hence, despite the fact that α, β_i, β_i' and γ are basis elements, integral coefficients arise in the multiplication if α and β_i or β_i' and γ have common factors). Notice that if a denotes the basis element of $D_{p_1-u}(F)$ corresponding to $a_{u+1} \leqslant \cdots \leqslant a_{p_1}$, then $\beta_i = a$ for precisely one i. For that choice of i, then, α and β_i have no common factors (since $a_u < a_{u+1}$), and β_i' and γ have no common factors (since $b_h < b_{h+1}$). Thus the tableau $\bar{T}$ occurs only once, with coefficient ± 1, in the sum $\sum \alpha\beta_i \otimes \beta_i'\gamma$ and, proceeding as in the proof of II.2.15, we obtain this slightly modified statement of that result:

LEMMA II.3.15. *If T is a co-row-standard tableau which is not standard, then there exist co-row-standard tableaux T_i with $T_i < T$ such that*

$$X_T - \sum c_i X_{T_i} \in \operatorname{Im} \square_{\lambda/\mu}',$$

where c_i are integers. In particular,

$$d_{\lambda/\mu}'(X_T) = \sum c_i d_{\lambda/\mu}'(X_{T_i}).$$

Note that in II.3.15 we have to allow for the integral coefficients c_i (coming from the multiplication in $D(F)$), but this in no way alters the applicability of II.3.15 to the proof of II.3.16. The rest of the proof of II.3.16 (i.e., linear independence) proceeds formally as in the proof of II.2.16.

II.4. *Duality and Decompositions*

Using the universal freeness of $L_{\lambda/\mu}F$ and $K_{\lambda/\mu}F$ and the dual nature of their definitions we obtain the following proposition:

PROPOSITION II.4.1. *Let F be a free R-module, and $F^* = \operatorname{Hom}_R(F, R)$ its dual. If $\mu \subset \lambda$ are partitions, then*

$$L_{\lambda/\mu}(F)^* \approx K_{\tilde{\lambda}/\tilde{\mu}}(F^*).$$

The canonical isomorphism between $\bar{L}_{\lambda/\mu}(F)$ and $L_{\lambda/\mu}(F)$ yields:

PROPOSITION II.4.2. *Let F be a free module of rank n, and let $\mu=(\mu_1,\ldots,\mu_q) \subseteq \lambda=(\lambda_1,\ldots,\lambda_q)$ be partitions. We have $(\mu_1-\mu_q,\ldots,\mu_1-\mu_2,0) \subseteq (n+\mu_1-\lambda_q,\ldots,n+\mu_1-\lambda_1)$, and we denote by $(\lambda/\mu)^*$ the skew-diagram associated to this latter pair of partitions. Then there is a natural isomorphism*

$$L_{\lambda/\mu}F \otimes \underbrace{\Lambda^n F^* \otimes \cdots \otimes \Lambda^n F^*}_{q} \to L_{(\lambda/\mu)^*}(F^*)$$

induced by tensor products of natural isomorphisms $\theta\colon \Lambda^p F \otimes \Lambda^n F^* \to \Lambda^{n-p} F^*$ *on the respective generators and relations.*

Proof. We will prove the proposition in the case where $q=2$, from which the general case follows. For convenience, set $p_i = \lambda_i - \mu_i$. The proof is based on the commutativity of the following diagram of generators and relations:

$$\begin{array}{ccc} \sum\limits_{t=\mu_1-\mu_2+1}^{p_2} \Lambda^{p_1+t}F \otimes \Lambda^{p_2-t}F \otimes \Lambda^n F^* \otimes \Lambda^n F^* & \xrightarrow{\square\otimes 1} & \Lambda^{p_1}F \otimes \Lambda^{p_2}F \otimes \Lambda^n F^* \otimes \Lambda^n F^* \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ \sum\limits_{t=\mu_1-\mu_2+1}^{p_2} \Lambda^{n-p_2+t}F^* \otimes \Lambda^{n-p_1-t}F^* & \xrightarrow{\quad\square\quad} & \Lambda^{n-p_2}F^* \otimes \Lambda^{n-p_1}F^*. \end{array}$$

Observe that the cokernels of the horizontal maps are $L_{\lambda/\mu}F \otimes \Lambda^n F^* \otimes \Lambda^n F^*$ and $L_{(\lambda/\mu)^*}(F^*)$. Since the vertical maps are isomorphisms, there is an induced isomorphism on the cokernels, as desired. It only remains to check the commutativity of the diagram, for which we introduce some notation.

Choose a basis $\{e_1, \dots, e_n\}$ for F and a dual basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ for F^*. For any subset $I = \{i_1 < \cdots < i_p\}$ of $\{1, \dots, n\}$, we denote the element $e_{i_1} \wedge \cdots \wedge e_{i_p}$ by e_I (and the same for the ε_i). For simplicity, we write ε for $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$. In this notation, the isomorphism $\theta\colon \Lambda^p F \otimes \Lambda^n F^* \to \Lambda^{n-p} F$ takes $e_I \otimes \varepsilon$ to $e_I(\varepsilon) = \pm\, \varepsilon_{I^*}$, where I^* is the complement of I in $\{1, \dots, n\}$. Recall from I.1.6 that if $\xi \in \Lambda F^*$, $x \in \Lambda F$, then $x(\xi) = n(x, \xi)$, where $n\colon \Lambda F \otimes \Lambda F^* \to \Lambda F^*$ is the natural ΛF-module structure on ΛF^*. On one side of the diagram we have

$$\theta((\square \otimes 1)(x_1 \otimes x_2 \otimes \varepsilon \otimes \varepsilon)) = \theta\left(\sum_{|I|=t} \varepsilon_I(x_1) \otimes e_I \wedge x_2 \otimes \varepsilon \otimes \varepsilon \right)$$

$$= \sum_{|I|=t} (e_I \wedge x_2)(\varepsilon) \otimes (\varepsilon_I(x_1))(\varepsilon).$$

On the other side,

$$\square(\theta(x_1 \otimes x_2 \otimes \varepsilon \otimes \varepsilon)) = \square(x_2(\varepsilon) \otimes x_1(\varepsilon)) = \sum_{|I|=t} e_I(x_2(\varepsilon)) \otimes \varepsilon_I \wedge (x_1(\varepsilon)).$$

Since $e_I(x_2(\varepsilon)) = (e_I \wedge x_2)(\varepsilon)$, it suffices to check that $(\varepsilon_I(x_1))(\varepsilon) = \varepsilon_I \wedge (x_1(\varepsilon))$. This follows from the following formula, which is Corollary 1.3 in [4]:

$$(\lambda(x))(\alpha) = \sum_i (-1)^{(1+\deg x)\,\lambda_i'} \lambda_i \wedge (x(\lambda_i' \wedge \alpha)),$$

where λ, $\alpha \in \Lambda F^*$, $x \in \Lambda F$ are homogeneous elements and $\Delta(\lambda) = \sum_i \lambda_i \otimes \lambda_i'$.

The remainder of this section is devoted to the proof of a decomposition theorem for $L_{\lambda/\mu}(F \oplus G)$. The aim is to put a filtration on $L_{\lambda/\mu}(F \oplus G)$ whose associated graded module is $\sum_{\mu \subseteq \nu \subseteq \lambda} L_{\lambda/\mu}(F) \otimes L_{\lambda/\mu}(G)$. In order to do this, we must introduce a total order on partitions.

DEFINITION II.4.3. Let λ and μ be partitions in $\mathbb{N}^\infty$. Then $\lambda \geqslant \mu$ if, under the lexicographic order induced by the usual total ordering of $\mathbb{N}$, λ is greater than or equal to μ.

Thus, if $\lambda = (\lambda_1, \ldots, \lambda_q)$, $\mu = (\mu_1, \ldots, \mu_q)$, then $\lambda \geqslant \mu$ if $\lambda_1 = \mu_1, \ldots, \lambda_i = \mu_i$, and $\lambda_{i+1} \geqslant \mu_{i+1}$.

This total ordering of partitions is clearly consistent with the partial ordering of inclusion. That is, if $\mu \subset \lambda$, then clearly $\mu \leqslant \lambda$.

If $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_{m+n}\}$ are bases for F and G, and $\mu \subset \lambda$ are partitions, then $X = \{x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}\}$ is a basis for $F \oplus G$, and we may consider $\mathrm{Tab}_{\lambda/\mu}(X)$, where X is totally ordered as indicated by the subscripts. If $T \in \mathrm{Tab}_{\lambda/\mu}(X)$, denote by $\eta(T)$ the sequence in $\mathbb{N}^\infty$ whose ith coordinate is $\mu_i +$ the number of basis elements of F in the ith row of T. Notice that if T is a standard tableau, then $\eta(T)$ is a partition, and $\mu \subseteq \eta(T) \subseteq \lambda$.

LEMMA II.4.4. *Let S and $T \in \mathrm{Tab}_{\lambda/\mu}(X)$ with $S \leqslant T$ (under our pseudo-order of tableaux). Then $\eta(S) \geqslant \eta(T)$.*

Proof. Assume that $\eta(S) \neq \eta(T)$ and let k be the first integer such that $\eta_k(S) \neq \eta_k(T)$. We want to show that $\eta_k(S) > \eta_k(T)$.

Since $S \leqslant T$, we know that $S_{p,q} \geqslant T_{p,q}$ for all p, q. Notice that

$$S_{i,m} = \sum_{j=1}^{i} \eta_j(S) - \mu_j$$

and

$$T_{i,m} = \sum_{j=1}^{i} \eta_j(T) - \mu_j.$$

Therefore, since $\eta_i(S) = \eta_i(T)$ for all $i < k$, we have

$$S_{i,m} = T_{i,m} \qquad \text{for all } i < k.$$

But

$$S_{k,m} = \sum_{j=1}^{k-1} \eta_j(S) - \mu_j + \eta_k(S) - \mu_k,$$

$$T_{k,m} = \sum_{j=1}^{k-1} \eta_j(T) - \mu_j + \eta_k(T) - \mu_k$$

so that, since $S_{k,m} \neq T_{k,m}$ [because $\eta_k(S) \neq \eta_k(T)$] and $S_{k,m} \geqslant T_{k,m}$, we must have $S_{k,m} > T_{k,m}$, i.e., $\eta_k(S) > \eta_k(T)$.

Putting Lemma II.2.15 and the standard basis Theorem II.2.16 together with the above, we have immediately:

PROPOSITION II.4.5. *Let* $T \in \mathrm{Tab}_{\lambda/\mu}(S)$ *be row-standard but not standard. Then there exist* T_i *in* $\mathrm{Tab}_{\lambda/\mu}(S)$ *with* $T_i < T$ *such that* $d_{\lambda/\mu}(X_{T_i}) = \sum c_i d_{\lambda/\mu}(X_{T_i})$ *with* $c_i \in \mathbb{Z}$ *and* $\eta(T_i) \geqslant \eta(T)$.

PROPOSITION II.4.6. *Let* $T \in \mathrm{Tab}_{\lambda/\mu}(S)$. *Then there exist unique standard tableaux* T_i, *and unique integers* $c_i \neq 0$, *such that* $d_{\lambda/\mu}(X_T) = \sum c_i d_{\lambda/\mu}(X_{T_i})$ *and* $\eta(T_i) \geqslant \eta(T)$.

DEFINITION II.4.7. Let F and G be free modules, and $\mu \subset \gamma \subset \lambda$ partitions. Define submodules $M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G))$ and $\dot{M}_\gamma(\Lambda_{\lambda/\mu}(F \oplus G))$ as follows:

$$M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G)) = \mathrm{Image}\left(\sum_{\substack{\mu \subseteq \sigma \lessdot \lambda \\ \sigma \geqslant \gamma}} \Lambda_{\sigma/\mu}(F) \otimes \Lambda_{\lambda/\sigma}(G) \to \Lambda_{\lambda/\mu}(F \oplus G)\right);$$

$$\dot{M}_\gamma(\Lambda_{\lambda/\mu}(F \oplus G)) = \mathrm{Image}\left(\sum_{\substack{\mu \subseteq \sigma \subseteq \lambda \\ \sigma > \gamma}} \Lambda_{\sigma/\mu}(F) \otimes \Lambda_{\lambda/\sigma}(G) \to \Lambda_{\lambda/\mu}(F \oplus G)\right).$$

The maps indicated above are those obtained by tensoring the maps

$$\Lambda^{\sigma_i - \mu_i} F \otimes \Lambda^{\lambda_i - \sigma_i} G \to \Lambda^{\lambda_i - \mu_i}(F \oplus G).$$

DEFINITION II.4.8. Define $M_\gamma(L_{\lambda/\mu}(F \oplus G))$ and $\dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G))$ to be $d_{\lambda/\mu}(M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G)))$ and $d_{\lambda/\mu}(\dot{M}_\gamma(\Lambda_{\lambda/\mu}(F \oplus G)))$, respectively.

It follows immediately from Proposition II.4.6 that $\{d_{\lambda/\mu}(X_T)/T \in \mathrm{Tab}_{\lambda/\mu}(S),\ \eta(T) \geqslant \gamma\}$ spans $M_\gamma(L_{\lambda/\mu}(F \oplus G))$ and that $\{d_{\lambda/\mu}(X_T)/T$ standard in $\mathrm{Tab}_{\lambda/\mu}(S),\ \eta(x) \geqslant \gamma\}$ is an R-basis for $M_\gamma(L_{\lambda/\mu}(F \oplus G))$.

PROPOSITION II.4.9. *The map* $\Lambda_{\gamma/\mu}(F) \otimes \Lambda_{\lambda/\gamma}(G) \to^{\psi} M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G))$ *induces a map*

$$\bar{\psi}_\gamma \colon L_{\gamma/\mu}(F) \otimes L_{\lambda/\gamma}(G) \to M_\gamma(L_{\lambda/\mu}(F \oplus G))/\dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G)).$$

Proof. Since $L_{\gamma/\mu}(F) = \operatorname{coker} \square_{\gamma/\mu}$ and $L_{\lambda/\gamma}(G) = \operatorname{coker} \square_{\lambda/\gamma}$ (see II.2.10 for definition of $\square_{\gamma/\mu}$), it suffices to prove that $\psi(\operatorname{Im}(\square_{\gamma/\mu}) \otimes \Lambda_{\lambda/\gamma}(G) + \Lambda_{\gamma/\mu}(F) \otimes \operatorname{Im}(\square_{\lambda/\gamma}))$ is contained in $N = M_\gamma(\Lambda_{\lambda/\mu}(F \oplus G)) + \operatorname{Im} \square_{\lambda/\mu}$. We will first show that $\psi(\operatorname{Im}(\square_{\gamma/\mu}) \otimes \Lambda_{\lambda/\gamma}(G))$ is contained in N.

Recall that $\square_{\gamma/\mu}$ is a sum of maps

$$\sum_{i=1}^{q-1} \sum_{l=\mu_i-\mu_{i+1}+1}^{\gamma_{i+1}-\mu_{i+1}} \Lambda^{\gamma_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\gamma_i-\mu_i+l}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}-l}F \otimes \cdots \otimes \Lambda^{\gamma_q-\mu_q}F \to \Lambda_{\gamma/\mu}(F)$$

so that, if we fix i and l, it suffices to show that $\psi(\operatorname{Im}(\Lambda^{\gamma_1-\mu_1}F \otimes \cdots \otimes \Lambda^{\gamma_i-\mu_i+l}F \otimes \Lambda^{\gamma_{i+1}-\mu_{i+1}-l}F \otimes \cdots \otimes \Lambda^{\gamma_q-\mu_q}F) \otimes \Lambda_{\lambda/\mu}(G))$ is contained in N.

If we take basis elements x_{I_k} of $\Lambda^{\gamma_k-\mu_k}F$ for $k \neq i, i+1$, and $x_{I_i}, x_{I_{i+1}}$ basis elements of $\Lambda^{\gamma_i-\mu_i+l}F$ and $\Lambda^{\gamma_{i+1}-\mu_{i+1}-l}F$, respectively, then $\psi(\operatorname{Im}(x_{I_1} \otimes \cdots \otimes x_{I_i} \otimes x_{I_{i+1}} \otimes \cdots \otimes x_{I_q}) \otimes y_{J_1} \otimes \cdots \otimes y_{J_q}) = \sum_U \pm x_{I_1} \wedge y_{J_1} \otimes \cdots \otimes x_{I_{i-1}} \wedge y_{J_{i-1}} \otimes x_U \wedge y_{J_i} \otimes x_{U'} \wedge x_{I_{i+1}} \wedge y_{J_{i+1}} \otimes \cdots \otimes x_{I_q} \wedge y_{J_q}$, where y_{J_k} are basis elements of $\Lambda^{\lambda_k-\gamma_k}G$, U runs over all subsets of order $\gamma_i - \mu_i$ of the index set I_i, and U' is the complement of U in I_i (hence, U' is of order l). Each summand in the above expression is $\pm X_{T_U}$, where $T_U \in \operatorname{Tab}_{\lambda/\mu}(S)$ is the tableau associated to the tensor product of basis elements of $\Lambda^{\lambda_k-\mu_k}(F \oplus G)$ in the usual way. Clearly, $\eta(T_U) = \gamma$ for all γ (as it must be).

Now take, in $\Lambda^{\lambda_1-\mu_1}(F \oplus G) \otimes \cdots \otimes \Lambda^{\lambda_i-\mu_i+l}(F \oplus G) \otimes \Lambda^{\lambda_{i+1}-\mu_{i+1}-l}(F \oplus G) \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q}(F \oplus G)$, the basis element $x_{I_1} \wedge y_{J_1} \otimes \cdots \otimes x_{I_i} \wedge y_{J_i} \otimes x_{I_{i+1}} \wedge y_{J_{i+1}} \otimes \cdots \otimes x_{I_q} \wedge y_{J_q}$. Since $l > \mu_i - \mu_{i+1}$, we may apply the map $\square_{\lambda/\mu}$ to this element, obtaining

$$\sum_{w_1, w_2} \pm x_{I_1} \wedge y_{J_1} \otimes \cdots \otimes x_{w_1} \wedge y_{w_2} \otimes x_{w_1'} \wedge x_{I_{i+1}} \wedge y_{w_2'} \wedge y_{J_{i+1}} \otimes \cdots \otimes X_{I_q} \wedge y_{J_q},$$

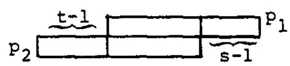
where w_1, w_2 are subsets of I_i and J_i whose orders add up to $\lambda_i - \mu_i$, and w_1' (w_2') is the complement of w_1 (w_2) in I_i (J_i). Since J_i is of order $\lambda_i - \gamma_i$, the subsets w_1 must be of order $\geqslant \gamma_i - \mu_i$. The sum above therefore breaks up into

$$\sum \pm X_{T_U} + \sum_{\bar{w}_1, \bar{w}_2} \pm x_{I_1} \wedge y_{J_1} \otimes \cdots \otimes x_{\bar{w}_1} \wedge y_{\bar{w}_2} \otimes x_{\bar{w}_1'} \wedge x_{I_{i+1}} \wedge y_{\bar{w}_2'} \wedge y_{J_{i+1}} \otimes \cdots \otimes x_{I_q} \wedge y_{J_q},$$

where $(\bar{w}_1, \bar{w}_2)$ runs through the pairs of subsets (w_1, w_2) described above with the order of $\bar{w}_1 > \gamma_i - \mu_i$. The corresponding tableaux $T_{\bar{w}_1, \bar{w}_2}$ then have the property that $\eta(T_{\bar{w}_1, \bar{w}_2}) > \gamma$, and (making suitable adjustment of sign) we have our desired result.

The proof that $\psi(\Lambda_{\gamma/\mu}(F) \otimes \operatorname{Im}(\square_{\lambda/\gamma}))$ is contained in N proceeds in a similar fashion, once we prove the following lemma.

LEMMA II.4.10. *Let p_1, p_2, s and t be positive integers, and consider the two-rowed skew-diagram represented by the following:*



.

Then the image of the map

$$\square\colon \sum_{l_1=t}^{p_2} \Lambda^{p_1+l_1}F \otimes \Lambda^{p_2-l_1}F \to \Lambda^{p_1}F \otimes \Lambda^{p_2}F$$

is equal to the image of the map

$$\tilde{\square}\colon \sum_{l_2=s}^{p_1} \Lambda^{p_1-l_1}F \otimes \Lambda^{p_2+l_2}F \to \Lambda^{p_1}F \otimes \Lambda^{p_2}F,$$

where $\square$ and $\tilde{\square}$ are obtained by the suitable diagonalizations and multiplications.

Proof. If $k+1$ is the number of overlaps in the above diagram, we have $t+k=p_2$ and $s+k=p_1$. Thus, if $s \leqslant l_2 \leqslant p_1$, then $0 \leqslant p_1 - l_2 \leqslant p_1 - s = k$. We may therefore apply Lemma II.2.9 with $u = p_1 - l_2, k = k+1$, $l = p_1 - l_2$, to obtain the fact that $\operatorname{Im} \tilde{\square} \subset \operatorname{Im} \square$. A completely symmetric argument (turn the picture through 180°), gives the reverse inclusion.

This lemma permits us to complete the proof of II.4.9 by simply replacing $\square_{\lambda/\mu}$ and $\square_{\lambda/\mu}$ by the appropriate maps $\tilde{\square}_{\lambda/\mu}$ and $\tilde{\square}_{\lambda/\mu}$ introduced in II.4.10.

THEOREM II.4.11. *The map $\bar{\psi}_\gamma\colon L_{\gamma/\mu}(F) \otimes L_{\lambda/\gamma}(G) \to M_\gamma(L_{\lambda/\mu}(F \oplus G))/\dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G))$ is an isomorphism. Hence, the modules $\{M_\gamma(L_{\lambda/\mu}(F \oplus G))/\mu \subseteq \gamma \subseteq \lambda\}$ give a filtration of $L_{\lambda/\mu}(F \oplus G)$ whose associated graded module is isomorphic to $\sum_{\mu \subseteq \gamma \subseteq \lambda} L_{\gamma/\mu}(F) \otimes L_{\lambda/\gamma}(G)$.*

Proof. The only fact that remains to be verified is that $\bar{\psi}_\gamma$ is an isomorphism. But the standard basis theorem for $L_{\gamma/\mu}(F)$ and $L_{\lambda/\gamma}(G)$ and the definition of the map $\bar{\psi}_\gamma$ clearly show that $\bar{\psi}_\gamma$ carries a basis of $L_{\gamma/\mu}(F) \otimes L_{\lambda/\gamma}(G)$ (i.e., the tensor product of the standard bases) bijectively onto a basis for $M_\gamma(L_{\lambda/\mu}(F \oplus G))/\dot{M}_\gamma(L_{\lambda/\mu}(F \oplus G))$.

There is an analogous decomposition theorem for coSchur functors and both decomposition theorems are special cases of Theorem V.1.13 for Schur complexes.

III. Cauchy Decomposition Formulas for Schur Functors

III.1. *The Decomposition of the Symmetric Algebra $S(F \otimes G)$*

Let F, G be finitely generated free R-modules. The symmetric algebra $S(F \otimes G)$ is naturally a $GL(F) \times GL(G)$-module. If R contains $\mathbb{Q}$ we have the direct sum decomposition $S(F \otimes G) = \sum L_\lambda F \otimes L_\lambda G$ into irreducible $GL(F) \times GL(G)$-modules (where λ runs over all partitions). In general, the above decomposition holds only up to filtration and the terms $L_\lambda F \otimes L_\lambda G$ need not be irreducible. In this section we will construct a universal filtration of $S_k(F \otimes G)$ whose associated graded object is $\sum_{|\lambda| = k} L_\lambda F \otimes L_\lambda G$.

The cases $k = 0, 1$ being trivial, we begin by examining $S_2(F \otimes G)$ in detail. The only partitions of weight 2 are (2) and (1, 1). The corresponding terms of the decomposition are $\Lambda^2 F \otimes \Lambda^2 G$ and $S_2 F \otimes S_2 G$. To see this we start with the natural embedding $\Lambda^2 F \otimes \Lambda^2 G \to S_2(F \otimes G)$ which sends $f_1 \wedge f_2 \otimes g_1 \wedge g_2$ to the determinant

$$\begin{vmatrix} f_1 \otimes g_1 & f_1 \otimes g_2 \\ f_2 \otimes g_1 & f_2 \otimes g_2 \end{vmatrix} = (f_1 \otimes g_1)(f_2 \otimes g_2) - (f_1 \otimes g_2)(f_2 \otimes g_1).$$

Identifying $\Lambda^2 F \otimes \Lambda^2 G$ with its image in $S_2(F \otimes G)$, we get $\Lambda^2 F \otimes \Lambda^2 G$ as the first piece of the filtration. Now we have to show $S_2(F \otimes G)/\Lambda^2 F \otimes \Lambda^2 G$ is isomorphic to $S_2 F \otimes S_2 G$. Consider the natural projection $F \otimes F \otimes G \otimes G \to S_2(F \otimes G)$ sending $f_1 \otimes f_2 \otimes g_1 \otimes g_2 \to (f_1 \otimes g_1)(f_2 \otimes g_2)$. Clearly this map sends $\Lambda^2 F \otimes G \otimes G$ and $F \otimes F \otimes \Lambda^2 G$ into $\Lambda^2 F \otimes \Lambda^2 G$. For any free R-module M, the quotient $M \otimes M/\Lambda^2 M$ is $S_2 M$ (here we are viewing $\Lambda^2 M$ as a submodule of $M \otimes M$ via the canonical injection $\Delta: \Lambda^2 M \to M \otimes M$ sending $m_1 \wedge m_2$ to $m_1 \otimes m_2 - m_2 \otimes m_1$). It follows that the quotient $F \otimes F \otimes G \otimes G/(\Lambda^2 F \otimes G \otimes G + F \otimes F \otimes \Lambda^2 G)$ is $S_2 F \otimes S_2 G$. Since the projection $F \otimes F \otimes G \otimes G \to S_2(F \otimes G)$ sends $\Lambda^2 F \otimes G \otimes G + F \otimes F \otimes \Lambda^2 G$ into $\Lambda^2 F \otimes \Lambda^2 G$ we have an induced epimorphism on the quotients β: $S_2 F \otimes S_2 G \to S_2(F \otimes G)/\Lambda^2 F \otimes \Lambda^2 G$. It is easy to check that $\text{rank}(\Lambda^2 F \otimes \Lambda^2 G) + \text{rank}(S_2 F \otimes S_2 G) = \text{rank}(S_2(F \otimes G))$ and this guarantees that β is an isomorphism.

For $S_3(F \otimes G)$ there are three pieces of the filtration corresponding to the three partitions (3), (2, 1), (1, 1, 1) of weight 3 and the respective terms of the decomposition are $\Lambda^3 F \otimes \Lambda^3 G$, $L_{(2,1)} F \otimes L_{(2,1)} G$, $S_3 F \otimes S_3 G$. Define the first piece $M_{(3)}$ of the filtration to be the image of the natural embedding

$\Lambda^3 F \otimes \Lambda^3 G \to S_3(F \otimes G)$ which maps $f_1 \wedge f_2 \wedge f_3 \otimes g_1 \wedge g_2 \wedge g_3$ to the determinant

$$\begin{vmatrix} f_1 \otimes g_1 & f_1 \otimes g_2 & f_1 \otimes g_3 \\ f_2 \otimes g_1 & f_2 \otimes g_2 & f_2 \otimes g_3 \\ f_3 \otimes g_1 & f_3 \otimes g_2 & f_3 \otimes g_3 \end{vmatrix}.$$

The next piece $M_{(2,1)}$ is defined to be the sum of $M_{(3)}$ with the image of the map $\Lambda^2 F \otimes F \otimes \Lambda^2 G \otimes G \to S_3(F \otimes G)$ that takes $f_1 \wedge f_2 \otimes f_3 \otimes g_1 \wedge g_2 \otimes g_3$ to $((f_1 \otimes g_1)(f_2 \otimes g_2) - (f_1 \otimes g_2)(f_2 \otimes g_1))(f_3 \otimes g_3)$. Finally, the last piece $M_{(1,1,1)}$ is all of $S_3(F \otimes G)$. So we have a filtration $0 \subseteq M_{(3)} \subseteq M_{(2,1)} \subseteq M_{(1,1,1)} = S_3(F \otimes G)$ and $M_3 \cong \Lambda^3 F \otimes \Lambda^3 G$. We want to show $M_{(2,1)}/M_{(3)} \cong L_{(2,1)}F \otimes L_{(2,1)}G$. Identifying $\Lambda^3 F$ with the image of the embedding $\Delta: \Lambda^3 F \to \Lambda^2 F \otimes F$ we can think of $L_{(2,1)}F$ as the quotient $\Lambda^2 F \otimes F/\Lambda^3 F$. We claim that the map $\Lambda^2 F \otimes F \otimes \Lambda^2 G \otimes G \to S_3(F \otimes G)$ takes $\Lambda^3 F \otimes \Lambda^2 G \otimes G \oplus \Lambda^2 F \otimes F \otimes \Lambda^3 G$ into $M_{(3)}$. But this is clear as $f_1 \wedge f_2 \wedge f_3 \otimes g_1 \wedge g_2 \wedge g_3$ is mapped to

$$\sum_{\sigma(1)<\sigma(2)} \operatorname{sgn}(\sigma) \begin{vmatrix} f_{\sigma(1)} \otimes g_1 & f_{\sigma(1)} \otimes g_2 \\ f_{\sigma(2)} \otimes g_1 & f_{\sigma(2)} \otimes g_2 \end{vmatrix} \cdot (f_{\sigma(3)} \otimes g_3),$$

which is nothing but the Laplace expansion of

$$\begin{vmatrix} f_1 \otimes g_1 & f_1 \otimes g_2 & f_1 \otimes g_3 \\ f_2 \otimes g_1 & f_2 \otimes g_2 & f_2 \otimes g_3 \\ f_3 \otimes g_1 & f_3 \otimes g_2 & g_3 \otimes g_3 \end{vmatrix} \in M_{(3)}.$$

We have therefore an induced epimorphism $\beta_{(2,1)}: L_{(2,1)}F \otimes L_{(2,1)}G \to M_{(2,1)}/M_{(3)}$. Similarly we can construct an epimorphism $\beta_{(1,1,1)}: S_3 F \otimes S_3 G \to M_{(1,1,1)}/M_{(2,1)}$. To prove that these are isomorphisms one needs only to check that $\sum_{|\lambda|=3} \operatorname{rank}(L_\lambda F \otimes L_\lambda G) = \operatorname{rank} S_3(F \otimes G)$.

We now turn to the general case. Define a natural pairing $\langle\, , \rangle: \Lambda^p F \otimes \Lambda^p G \to S_p(F \otimes G)$ of R-modules by mapping $f_1 \wedge \cdots \wedge f_p \otimes g_1 \wedge \cdots \wedge g_p$ to the p-by-p determinant $(-1)^{p(p-1)/2} \sum \operatorname{sgn}(\sigma)(f_{\sigma(1)} \otimes g_1) \cdots (f_{\sigma(p)} \otimes g_p)$ which we denote by $\langle f_1 \wedge \cdots \wedge f_p, g_1 \wedge \cdots \wedge g_p \rangle$. Extend this inductively to a pairing $\langle\, , \rangle: \Lambda_\lambda F \otimes \Lambda_\lambda G \to S_k(F \otimes G)$ where $\lambda = (\lambda_1, ..., \lambda_t)$ is a partition of weight k. If $t > 1$ let $\lambda' = (\lambda_1, ..., \lambda_{t-1})$ and $k' = |\lambda'|$. Observing that $\Lambda_\lambda F = \Lambda_{\lambda'} F \otimes \Lambda^{\lambda_t} F$ we take $\langle\, , \rangle: \Lambda_\lambda F \otimes \Lambda_\lambda G \to S_k(F \otimes G)$ to be the composition $\Lambda_{\lambda'} F \otimes \Lambda^{\lambda_t} F \otimes \Lambda_{\lambda'} G \otimes \Lambda^{\lambda_t} G \to \Lambda_{\lambda'} F \otimes \Lambda_{\lambda'} G \otimes \Lambda^{\lambda_t} F \otimes \Lambda^{\lambda_t} G \to S_{k'}(F \otimes G) \otimes S_{\lambda_t}(F \otimes G) \to S_k(F \otimes G)$. We let $\langle \Lambda_\lambda F, \Lambda_\lambda G \rangle$ denote the image of the pairing $\langle\, , \rangle: \Lambda_\lambda F \otimes \Lambda_\lambda G \to S_k(F \otimes G)$. More generally, if V, W are submodules of $\Lambda_\lambda F$, $\Lambda_\lambda G$, respectively, we let $\langle V, W \rangle$ denote the image of the composition $V \otimes W \to \Lambda_\lambda F \otimes \Lambda_\lambda G \to S_k(F \otimes G)$.

We define a natural filtration $\{M_\lambda(S_k(F \otimes G)) \mid |\lambda| = k\}$ of $S_k(F \otimes G)$ by

$M_\lambda(S_k(F\otimes G))=\sum_{\gamma\geqslant\lambda,|\gamma|=k}\langle\Lambda_\lambda F,\Lambda_\gamma G\rangle$. For convenience we shall write M_λ for $M_\lambda(S_k(F\otimes G))$ and $\dot{M}_\lambda$ for $\sum_{\gamma>\lambda,|\gamma|=k}\langle\Lambda_\gamma F,\Lambda_\gamma G\rangle$. Since $\langle F\otimes\cdots\otimes F, G\otimes\cdots\otimes G\rangle=S_k(F\otimes G)$ we have a filtration

$$0\subseteq M_{(k)}\subseteq M_{(k-1,1)}\subseteq\cdots\subseteq M_{(1,\dots,1)}=S_k(F\otimes G)$$

induced by the lexicographic order $\geqslant$ on partitions of weight k. Observe that the first piece of the filtration, $M_{(k)}$, is isomorphic to $\Lambda^kF\otimes\Lambda^kG$. Our goal is to show that the associated graded object of the above filtration is $\sum_{|\lambda|=k}L_\lambda F\otimes L_\lambda G$, i.e., to show $M_\lambda/\dot{M}_\lambda\cong L_\lambda F\otimes L_\lambda G$.

In dealing with the filtration $\{M_\lambda\}$ it is convenient to use double tableaux. Let $X=\{x_1,\dots,x_m\}$ and $Y=\{y_1,\dots,y_n\}$ be ordered bases for F and G, respectively. If $S\in\mathrm{Tab}_\lambda(X)$, $T\in\mathrm{Tab}_\lambda(Y)$, we let the double tableau $(S\mid T)$ of shape λ denote the element $\langle X_S, Y_T\rangle$ of $S_k(F\otimes G)$. If $\lambda=(\lambda_1,\dots,\lambda_p)$ let $S^i=(S(i,1),\dots,S(i,\lambda_i))$ be the ith row of S and similarly let T^i be the ith row of T. Then $(S\mid T)=\pm(S^1\mid T^1)\cdot(S^2\mid T^2)\cdots(S^p\mid T^p)$, where

$$(S^i\mid T^i)=\langle S(i,1)\wedge\cdots\wedge S(i,\lambda_i), T(i,1)\wedge\cdots\wedge T(i,\lambda_i)\rangle.$$

The module M_λ can be described as the R-linear span of all double tableaux of shape $\geqslant\lambda$ and weight k.

The first step in the proof that $L_\lambda F\otimes L_\lambda G\cong M_\lambda/\dot{M}_\lambda$ is the definition of a natural map $\beta_\lambda: L_\lambda F\otimes L_\lambda G\to M_\lambda/\dot{M}_\lambda$. We already have a map $\langle\ ,\ \rangle:\Lambda_\lambda F\otimes\Lambda_\lambda G\to M_\lambda$ from which we want to derive β_λ. Since $L_\lambda F=\Lambda_\lambda F/\square(\Lambda_\lambda F)$ all we have to show is $\langle\square(\Lambda_\lambda F),\Lambda_\lambda G\rangle+\langle\Lambda_\lambda F,\square(\Lambda_\lambda G)\rangle\subseteq\dot{M}_\lambda$.

PROPOSITION III.1.1. *Let $\bar{R}$ denote the R-algebra $S(F\otimes G)$ and let $\bar{F}=\bar{R}\otimes F$, $\bar{G}=\bar{R}\otimes G$ be induced $\bar{R}$-modules. There is a natural $\bar{R}$-pairing $\langle\ ,\ \rangle_\alpha:\Lambda\bar{F}\otimes_{\bar{R}}\Lambda\bar{G}\to\bar{R}$ induced by a map $\alpha:\Lambda\bar{F}\to\Lambda\bar{G}^*$ of $\bar{R}$-Hopf algebras. Moreover, if $\lambda=(\lambda_1,\dots,\lambda_p)$ is a partition of weight k the pairing $\langle\ ,\ \rangle$: $\Lambda_\lambda F\otimes_R\Lambda_\lambda G\to S_k(F\otimes G)$ defined above is the restriction of*

$$\langle\ ,\ \rangle_{\alpha\otimes\cdots\otimes\alpha}:\Lambda\bar{F}\otimes_{\bar{R}}\cdots\otimes_{\bar{R}}\Lambda\bar{F}\otimes_{\bar{R}}\Lambda\bar{G}\otimes_{\bar{R}}\cdots\otimes_{\bar{R}}\Lambda\bar{G}\to\bar{R}$$

to the R-submodule $\Lambda_\lambda F\otimes\Lambda_\lambda G$ (where $\alpha\otimes\cdots\otimes\alpha$ is the $\bar{R}$-Hopf algebra map $\alpha^{\otimes p}:(\Lambda\bar{F})^{\otimes p}\to(\Lambda\bar{G}^*)^{\otimes p}$).

Proof. We take α to be the map $\Lambda\phi:\Lambda\bar{F}\to\Lambda\bar{G}^*$ associated to the "generic" map $\phi:\bar{F}\to\bar{G}^*$ which is defined as follows. Let $\phi_0: F\to F\otimes G\otimes G^*$ be the R-map which sends $f\in F$ to $f\otimes c_G$, where $c_G\in G\otimes G^*$ is the trace element [the element corresponding to the identity map $1_G\in\mathrm{Hom}(G,G)$ under the canonical isomorphism $G\otimes G^*\cong\mathrm{Hom}(G,G)$].

ϕ is the extension of $\phi_0: F \to F \otimes G \otimes G^*$ to $\bar{F} \to \bar{G}^*$, i.e., ϕ is the composition

$$\bar{F} = S(F \otimes G) \otimes F \xrightarrow{1 \otimes \phi_0} S(F \otimes G) \otimes S_1(F \otimes G)$$
$$\otimes\, G^* \xrightarrow{m \otimes 1} S(F \otimes G) \otimes G^* = \bar{G}^*.$$

Since $\Lambda(-)$ is a functor from $\bar{R}$-modules to R-Hopf algebras, $\alpha = \Lambda\phi$ is a map of $\bar{R}$-Hopf algebras. The naturality of α follows from the naturality of ϕ_0 and the functoriality of $\Lambda(-)$. The second assertion of the proposition is a straightforward computation of the pairing $\langle\ ,\ \rangle_{\alpha \otimes \cdots \otimes \alpha}$.

COROLLARY III.1.2. $\langle \square(\Lambda_\lambda F), \Lambda_\lambda G\rangle + \langle \Lambda_\lambda F, \square(\Lambda_\lambda G)\rangle \subseteq \dot{M}_\lambda$.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$. We first consider the case $p = 2$. When $\lambda = (\lambda_1, \lambda_2)$, $\square(\Lambda_\lambda F)$ is the image of the map $\sum_{t=1}^{\lambda_2} \Lambda^{\lambda_1+t}F \otimes \Lambda^{\lambda_2-t}F \xrightarrow{\square} \Lambda^{\lambda_1}F \otimes \Lambda^{\lambda_2}F$. Let $a \in \Lambda^{\lambda_1+t}F$, $b \in \Lambda^{\lambda_2-t}F$, $c \in \Lambda^{\lambda_1}G$, $d \in \Lambda^{\lambda_2}G$. Let $\sum a_i^1 \otimes a_i^2$ be the image of a under the map $\Delta: \Lambda^{\lambda_1+t}F \to \Lambda^{\lambda_1}F \otimes \Lambda^t F$ and let $\sum d_j^1 \otimes d_j^2$ be the image of d under the map $\Delta: \Lambda^{\lambda_2}G \to \Lambda^t G \otimes \Lambda^{\lambda_2-t}G$. We want to show $\sum \langle a_i^1 \otimes a_i^2 \wedge b, c \otimes d\rangle \subseteq \dot{M}_\lambda$. Observing that $\square_{\Lambda F}$ is the restriction of $\square_{\Lambda \bar{F}}$ and using the proposition along with Proposition I.1.8 we have the following:

$$\sum \langle a_i^1 \otimes a_i^2 \wedge b, c \otimes d\rangle$$
$$= \langle \square(a \otimes b), c \otimes d\rangle = \langle a \otimes b, \tilde{\square}(c \otimes d)\rangle$$
$$= \sum \langle a \otimes b, c \wedge d_j^1 \otimes d_j^2\rangle \in \langle \Lambda^{\lambda_1+t}F \otimes \Lambda^{\lambda_2-t}F, \Lambda^{\lambda_1+t}G \otimes \Lambda^{\lambda_2-t}G\rangle$$

which is contained in $\dot{M}_\lambda$ because $(\lambda_1 + t, \lambda_2 - t) > (\lambda_1, \lambda_2)$.

Now suppose $p > 2$. Recall that $\square(\Lambda_\lambda F) = \sum_{s=1}^{p-1} \Lambda^{\lambda_1}F \otimes \cdots \otimes \Lambda^{\lambda_{s-1}}F \otimes \square(\Lambda_{(\lambda_s, \lambda_{s+1})}F) \otimes \Lambda^{\lambda_{s+2}}F \otimes \cdots \otimes \Lambda^{\lambda_p}F$. Using the case $p = 2$ we have $\langle \square(\Lambda_{(\lambda_s,\lambda_{s+1})}F), \Lambda_{(\lambda_s,\lambda_{s+1})}G\rangle \subseteq \sum_{t=1}^{\lambda_{s+1}} \langle \Lambda_{(\lambda_s+t,\lambda_{s+1}-t)}F, \Lambda_{(\lambda_s+t,\lambda_{s+1}-t)}G\rangle$. Let $\lambda(s, t)$ denote the partition obtained by rearranging the sequence $(\lambda_1, \ldots, \lambda_{s-1}, \lambda_s + t, \lambda_{s+1} - t, \lambda_{s+2}, \ldots, \lambda_p)$. It is clear that $\lambda(s, t) > \lambda$ because $t \geqslant 1$. Therefore $\langle \square(\Lambda_\lambda F), \Lambda_\lambda G\rangle \subseteq \sum_{s=1}^{p-1} \sum_{t=1}^{\lambda_{s+1}} \langle \Lambda_{\lambda(s,t)}F, \Lambda_{\lambda(s,t)}G\rangle \subseteq \dot{M}_\lambda$. The proof of $\langle \Lambda_\lambda F, \square(\Lambda_\lambda G)\rangle \subseteq \dot{M}_\lambda$ is completely symmetric.

COROLLARY III.1.3. *The natural map* $\langle\ ,\ \rangle: \Lambda_\lambda F \otimes \Lambda_\lambda G \to M_\lambda$ *induces a map* $\beta_\lambda: L_\lambda F \otimes L_\lambda G \to M_\lambda/\dot{M}_\lambda$. *Moreover, it follows readily from the definitions of* $M_\lambda, \dot{M}_\lambda$ *that* β_λ *is an epimorphism.*

THEOREM III.1.4 [The Standard Basis Theorem for $S(F \otimes G)$]. *The maps* $\beta_\lambda: L_\lambda F \otimes L_\lambda G \to M_\lambda/\dot{M}_\lambda$ *are isomorphisms and therefore the* *associated graded object of the filtration* $\{M_\lambda(S_k(F \otimes G))\}$ *is* $\sum_{|\lambda|=k} L_\lambda F \otimes L_\lambda G$.

Proof. We already observed that the β_λ are epimorphisms. Let $S \in \text{Tab}_\lambda(X)$, $T \in \text{Tab}_\lambda(Y)$. We say that the double tableau $(S \mid T)$ is standard if both S and T are standard. Let B_k denote the set of standard double tableaux of weight k. Since the standard tableaux generate $L_\lambda F, L_\lambda G$ (the Standard Basis Theorem for L_λ) the fact that the β_λ are isomorphisms implies that the standard double tableaux of shape $\geqslant \lambda$ and weight k generate M_λ. In particular, B_k generates $M_{(1,\ldots,1)} = S_k(F \otimes G)$.

Claim. If $L_\lambda F \otimes L_\lambda G \neq 0$, then the module $M_\lambda/\dot{M}_\lambda \neq 0$.

First we observe that the theorem follows from the claim. For if the claim is true and we take $R = \mathbb{Q}$, the β_λ are forced to be isomorphisms because the $L_\lambda F \otimes L_\lambda G$ are irreducible $GL(F) \times GL(G)$-modules when $R = \mathbb{Q}$ and the β_λ are already known to be epimorphisms. Therefore the equality rank $(S_k(F \otimes G)) = \sum_{|\lambda|=k} \text{rank}(L_\lambda F \otimes L_\lambda G)$ holds over $\mathbb{Q}$ (and hence over any ring R by the universal freeness of the L_λ and S_k). From the Standard Basis Theorem for L_λ we know that $\#(B_k) = \sum_{|\lambda|=k} \text{rank}(L_\lambda F \otimes L_\lambda G)$. But we already observed that the set B_k generates the free R-module $S_k(F \otimes G)$ so that B_k must be an R-basis of $S_k(F \otimes G)$. This, in turn, implies that $M_\lambda/\dot{M}_\lambda$ is a free R-module with the same rank as $L_\lambda F \otimes L_\lambda G$. Therefore the maps β_λ must be isomorphisms.

We now proceed to prove the claim. Suppose $L_\lambda F \otimes L_\lambda G \neq 0$. By the Standard Basis Theorem for L_λ, this is equivalent to $\lambda_1 \leqslant \min(m, n)$, where $m = \text{rank}(F)$, $n = \text{rank}(G)$. Let C_λ^X, C_λ^Y denote the canonical tableaux in $\text{Tab}_\lambda(X)$, $\text{Tab}_\lambda(Y)$, respectively, that is, $C_\lambda^X(i,j) = x_j$, $C_\lambda^Y(i,j) = y_j$. Then $(C_\lambda^X \mid C_\lambda^Y) = \pm\prod_{t=1}^{p} \langle x_1 \wedge \cdots \wedge x_{\lambda_t}, y_1 \wedge \cdots \wedge y_{\lambda_t} \rangle$ is clearly a non-zero element of M_λ. Moreover, it is easy to see that there is no standard tableau T in $\text{Tab}_\gamma(X)$ of shape $\gamma > \lambda$ such that $\text{content}(T) = \text{content}(C_\lambda^X)$. Since the double standard tableaux of shape $> \lambda$ and weight k span $\dot{M}_\lambda$, it follows that $(C_\lambda^X \mid C_\lambda^Y) \notin \dot{M}_\lambda$. Therefore $M_\lambda/\dot{M}_\lambda \neq 0$, proving the claim and the theorem.

III.2. *The Decomposition of the Exterior Algebra* $\Lambda(F \otimes G)$

Let F, G be finitely generated free R-modules. The exterior algebra $\Lambda(F \otimes G)$ is naturally a $GL(F) \times GL(G)$-module. If R contains $\mathbb{Q}$, then $\Lambda(F \otimes G)$ decomposes into the direct sum $\sum L_\lambda F \otimes L_{\tilde{\lambda}} G$ of irreducible $GL(F) \times GL(G)$-modules. This decomposition does not hold in general, not even up to filtration. There is, however, as we shall show in this section, a universal filtration of $\Lambda(F \otimes G)$ by $GL(F) \times GL(G)$-modules whose associated graded object is $\sum L_\lambda F \otimes K_\lambda G$. This is the correct generalization of the decomposition over $\mathbb{Q}$ because $L_{\tilde{\lambda}} G \cong K_\lambda G$, when R contains $\mathbb{Q}$.

We begin by defining a natural pairing $\langle\, ,\rangle \colon \Lambda^p F \otimes D_p G \to \Lambda^p(F \otimes G)$ by induction on p. For $p = 1$, we define $\langle f, g\rangle = f \otimes g$. For $p > 1$ we define

$$\langle f_1 \wedge \cdots \wedge f_p, g_1^{(\alpha_1)} \cdots g_t^{(\alpha_t)} \rangle$$
$$= \sum_{i=1}^{p} (-1)^{i-1} \langle f_i, g_1 \rangle \wedge \langle f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_p, g_1^{(\alpha_1 - 1)} g_2^{(\alpha_2)} \cdots g_t^{(\alpha_t)} \rangle$$

or, equivalently,

$$\sum_{i=1}^{t} \langle f_1, g_i \rangle \wedge \langle f_2 \wedge \cdots \wedge f_p, g_1^{(\alpha_1)} \cdots g_i^{(\alpha_i - 1)} \cdots g_t^{(\alpha_t)} \rangle,$$

where $\sum_{i=1}^{t} \alpha_i = p$ and $\alpha_i \geqslant 1$ for all i. Next we extend the above to a pairing $\langle \, , \rangle : \Lambda_\lambda F \otimes D_\lambda G \to \Lambda^k(F \otimes G)$, where $k = |\lambda|$. Let $\lambda = (\lambda_1, ..., \lambda_t)$ and define $\langle a_1 \otimes \cdots \otimes a_t, b_1 \otimes \cdots \otimes b_t \rangle = \langle a_1, b_1 \rangle \wedge \cdots \wedge \langle a_t, b_t \rangle$, where $a_i \in \Lambda^{\lambda_i} F$, $b_i \in D_{\lambda_i} G$.

We define a natural filtration $\{M_\lambda(\Lambda^k(F \otimes G)) \mid |\lambda| = k\}$ of $\Lambda^k(F \otimes G)$ by $M_\lambda(\Lambda^k(F \otimes G)) = \sum_{\gamma \geqslant \lambda, |\gamma| = k} \langle \Lambda_\gamma F, D_\gamma G \rangle$. For convenience, we shall write M_λ for $M_\lambda(\Lambda^k(F \otimes G))$ and $\dot{M}_\lambda$ for $\sum_{\gamma > \lambda, |\gamma| = k} \langle \Lambda_\gamma F, \Lambda_\gamma G \rangle$. Since $\langle F \otimes \cdots \otimes F, G \otimes \cdots \otimes G \rangle = \Lambda^k(F \otimes G)$, we have a filtration

$$0 \subseteq M_{(k)} \subseteq M_{(k-1,1)} \subseteq \cdots \subseteq M_{(1,\ldots,1)} = \Lambda^k(F \otimes G)$$

induced by the lexicographic order $\geqslant$ on partitions of weight k. We will show that $M_\lambda / \dot{M}_\lambda \cong L_\lambda F \otimes K_\lambda G$.

We shall again use double tableaux for convenience. Let $X = \{x_1, ..., x_m\}$, $Y = \{y_1, ..., y_n\}$ be ordered bases for F, G, respectively. If $S \in \mathrm{Tab}_\lambda(X)$, $T \in \mathrm{Tab}_\lambda(Y)$ we let the double tableau $(S \mid T)$ denote the element $\langle X_S, X_T \rangle$ of $\Lambda^k(F \otimes G)$. Let $S^i = (S(i, 1), ..., S(i, \lambda_i))$ be the ith row of S and T^i be the ith row of T. Then $(S \mid T) = (S^1 \mid T^1) \cdots (S^p \mid T^p)$. The module M_λ is the R-linear span of all double tableaux of shape $\geqslant \lambda$ and weight k.

Proposition III.2.1. *Let $\bar{R}$ denote the R-algebra $\Lambda(F \otimes G)$ and let $\bar{F} = \bar{R} \otimes F$, $\bar{G} = \bar{R} \otimes G$ be induced $\bar{R}$-modules. There is a natural $\bar{R}$-pairing $\langle \, , \rangle_\alpha : \Lambda \bar{F} \otimes_{\bar{R}} D\bar{G} \to \bar{R}$ induced by a map $\alpha : \Lambda \bar{F} \to S\bar{G}^*$ of $\bar{R}$-Hopf algebras. Moreover, if $\lambda = (\lambda_1, ..., \lambda_p)$ is a partition of weight k the pairing $\langle \, , \rangle$: $\Lambda_\lambda F \otimes_R D_\lambda G \to \Lambda^k(F \otimes G)$ defined above is the restriction of*

$$\langle \, , \rangle_{\alpha \otimes \cdots \otimes \alpha} : \Lambda \bar{F} \otimes_{\bar{R}} \cdots \otimes_{\bar{R}} \Lambda \bar{F} \otimes_{\bar{R}} D\bar{G} \otimes_{\bar{R}} \cdots \otimes_{\bar{R}} D\bar{G} \to \bar{R}$$

to the R-submodule $\Lambda_\lambda F \otimes \Lambda_\lambda G$.

Proof. $\bar{R} = \sum \bar{R}_k$ is a graded R-algebra where $\bar{R}_k = \Lambda^k(F \otimes G)$. The natural map $F \to F \otimes G \otimes G^* = \bar{R}_1 \otimes S_1 G^*$ which sends $f \in F$ to $f \otimes c_G$ induces an R-algebra map $\Lambda F \to \sum_k \bar{R}_k \otimes_R S_k G^*$ by the universal property of the exterior algebra (recall that $c_G \in G \otimes G^*$ is the trace element). This map extends canonically to an $\bar{R}$-algebra map $\Lambda \bar{F} \to S\bar{G}^*$. It is easy to check that $\alpha \circ \Delta_{\Lambda \bar{F}} = \Delta_{S\bar{G}^*} \circ \alpha$ so that α is a map of $\bar{R}$-Hopf algebras. Since $S\bar{G}^* \cong (D\bar{G})^*$ we have an induced pairing $\langle\, ,\, \rangle_\alpha : \Lambda \bar{F} \otimes_{\bar{R}} D\bar{G} \to \bar{R}$. It is a straightforward computation to check that $\langle\, ,\, \rangle : \Lambda_\lambda F \otimes D_\lambda G \to \Lambda^k(F \otimes G)$ is the restriction of $\langle\, ,\, \rangle_{\alpha \otimes \cdots \otimes \alpha}$ to $\Lambda_\lambda F \otimes D_\lambda G$.

COROLLARY III.2.2. $\langle \square(\Lambda_\lambda F), D_\lambda G\rangle + \langle \Lambda_\lambda F, \square(D_\lambda G) \subseteq \dot{M}_\lambda$.

Proof. Similar to the proof of Corollary III.1.2.

COROLLARY III.2.3. *The natural map* $\langle\, ,\, \rangle : \Lambda_\lambda F \otimes D_\lambda G \to M_\lambda$ *induces a map* $\beta_\lambda : L_\lambda F \otimes K_\lambda G \to M_\lambda / \dot{M}_\lambda$. *Moreover, it follows readily from the definitions of* $M_\lambda, \dot{M}_\lambda$ *that* β_λ *is an epimorphism.*

THEOREM III.2.4 [The Standard Basis Theorem for $\Lambda(F \otimes G)$]. *The maps* $\beta_\lambda : L_\lambda F \otimes K_\lambda G \to M_\lambda / \dot{M}_\lambda$ *are isomorphisms and therefore the associated graded object of the filtration* $\{M_\lambda(\Lambda^k(F \otimes G))\}$ *is* $\sum L_\lambda F \otimes K_\lambda G$.

Proof. We already observed that the β_λ are epimorphisms. Let $S \in \mathrm{Tab}_\lambda(X)$, $T \in \mathrm{Tab}_\lambda(Y)$. We say the double tableau $(S \mid T)$ is standard if S is standard and T is co-standard. Let B_K denote the set of standard double tableaux of weight k. Since the standard tableaux in $\mathrm{Tab}_\lambda(X)$ generate $L_\lambda F$ and the co-standard tableaux in $\mathrm{Tab}_\lambda(Y)$ generate $K_\lambda G$ (The Standard Basis Theorems for L_λ, K_λ) the fact that the β_λ are epimorphisms implies that the standard double tableaux of shape $\geqslant \lambda$ and weight k generate M_λ. In particular, B_k generates $\Lambda^k(F \otimes G)$.

Claim. If $L_\lambda F \otimes K_\lambda G \neq 0$ then the module $M_\lambda / \dot{M}_\lambda \neq 0$.

If the claim is true then the β_λ are isomorphisms when $R = \mathbb{Q}$ because the $L_\lambda F \otimes K_\lambda G$ are irreducible $GL(F) \times GL(G)$-modules when $R = \mathbb{Q}$. It follows that $\mathrm{rank}(S_k(F \otimes G)) = \sum_{|\lambda| = k} \mathrm{rank}(L_\lambda F \otimes K_\lambda G)$ and the latter number equals $\#(B_k)$. Therefore B_k must be an R-basis of $\Lambda^k(F \otimes G)$ forcing the β_λ to be isomorphisms.

To prove the claim, the condition $L_\lambda F \otimes K_\lambda G \neq 0$ is equivalent to $\lambda_1 \leqslant m = \mathrm{rank}(F)$ and $\tilde{\lambda}_1 \leqslant n = \mathrm{rank}(G)$ by the Standard Basis Theorems for L_λ, K_λ. Let $C_\lambda^X, C_{\tilde{\lambda}}^Y$ denote the canonical tableaux in $\mathrm{Tab}_\lambda(X)$, $\mathrm{Tab}_{\tilde{\lambda}}(Y)$, respectively, and let $D_\lambda^Y \in \mathrm{Tab}_\lambda(Y)$ be the transpose of $C_{\tilde{\lambda}}^Y$. So $C_\lambda^X(i,j) = X_j$ and $D_\lambda^Y(i,j) = x_i$. Then

$$\begin{aligned}(C_\lambda^X \mid D_\lambda^Y) &= \langle f_1 \wedge \cdots \wedge f_{\lambda_1} \otimes \cdots \otimes f_1 \wedge \cdots \wedge f_{\lambda_p}, g_1^{(\lambda_1)} \otimes \cdots \otimes g_p^{(\lambda_p)} \rangle \\ &= (f_1 \otimes g_1) \wedge \cdots \wedge (f_{\lambda_1} \otimes g_1) \wedge \cdots \wedge (f_1 \otimes g_p) \wedge \cdots \\ &\quad \wedge (f_{\lambda_p} \otimes g_p) \in M_\lambda\end{aligned}$$

is clearly non-zero. Moreover, $(C_\lambda^X \mid D_\lambda^Y) \notin \dot{M}_\lambda$ because there is no standard tableau T in $\mathrm{Tab}_\gamma(X)$ of shape $\gamma > \lambda$ such that $\mathrm{content}(T) = \mathrm{content}(C_\lambda^X)$ and the standard double tableaux of shape $>\lambda$ and weight k span $\dot{M}_\lambda$. It follows that $M_\lambda/\dot{M}_\lambda \neq 0$.

IV. The Littlewood–Richardson Rule for Schur Functors

IV.1. *The Schensted Process and Words of Yamanouchi*

The tensor product $L_\lambda F \otimes L_\mu F$ of two Schur functors decomposes into a direct sum of Schur functors, provided that the ground ring contains a field of characteristic zero. For example, $L_{(2,1)}F \otimes \Lambda^2 F = L_{(4,1)}F \oplus L_{(3,2)}F \oplus L_{(3,1,1)}F \oplus L_{(2,2,1)}F$. Notice that each Schur functor in this decomposition occurs only once. This, however, is not the case in the decomposition $L_{(2,1)}F \otimes L_{(2,1)}F = L_{(4,2)}F \oplus L_{(4,1,1)}F \oplus L_{(3,3)}F \oplus 2L_{(3,2,1)}F \oplus L_{(2,2,2)}F \oplus L_{(2,2,1,1)}F$ where the Schur functor $L_{(3,2,1)}F$ occurs twice. The integer 2 is called the multiplicity of $L_{(3,2,1)}F$ in $L_{(2,1)}F \otimes L_{(2,1)}F$. The Littlewood–Richardson rule provides an algorithm for not only determining which Schur functors appear in $L_\lambda F \otimes L_\mu F$ but for actually computing the multiplicities of these Schur functors. It should be noted that $L_\nu F$ cannot appear in $L_\lambda F \otimes L_\mu F$ unless $|\nu| = |\lambda| + |\mu|$ because $L_\nu F$ and $L_\lambda F \otimes L_\mu F$ are homogeneous polynomial functors of degrees $|\nu|$ and $|\lambda| + |\mu|$, respectively.

The statement and proof of the Littlewood–Richardson rule require a considerable amount of combinatorial machinery and therefore will be deferred until Section IV.2. We begin this section by describing a procedure known as the Schensted process.

Let λ be a partition, S a totally ordered set, U a standard tableau in $\mathrm{Tab}_\lambda(S)$, and $p \in S$. We define the "bumping of p into U" (denoted by $p \to U$) in the following recursive manner.

(1) Put p inside the top row of U in place of the least element p_1 in that row such that $p_1 \geqslant p$. If such a p_1 does not exist, adjoin a new box to the right of the first row of λ, place p in this box and stop. In this case we obtain the tableau $p \to U$ of shape $\lambda^1 = (\lambda_1 + 1, \lambda_2, \ldots, \lambda_q)$ and it is clear that $p \to U$ is standard.

(2) If p has replaced (or bumped) an element p_1 in the first row, then repeat step (1) by this time bumping p_1 into the second row.

Repeating this procedure we finally obtain a new tableau $p \to U$ of shape λ^1 where λ^1 is obtained from λ either by adjoining a new box to one of the nonempty rows of λ or by adjoining a new bottom row to λ consisting of a single box.

For example, if $p = 2$ and

$$U = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

then

$$p \to U = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

and the steps can be illustrated as follows:

$$2 \to \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = 3 \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = 3 \to \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

LEMMA IV.1.1. *The tableau $p \to U$ in* $\mathrm{Tab}_{\lambda^1}(S)$ *is standard.*

Proof. The row-standardness is obvious. To prove column-standardness, it is clearly sufficient to consider the case where U has two rows. So suppose that U is the tableau

u_1	...		...	u_r
v_1	...	v_m		

We already observed that $p \to U$ is standard if $p > u_r$. So suppose that p does bump some u_t $(t \leqslant r)$. Then $u_{t-1} < p \leqslant u_t$ and if u_t bumps some v_s in the second row we must have $s \leqslant t$ because $u_t \leqslant v_t$. The first column of $p \to U$ is (u_1, v_1, v_s) which is standard because $v_1 < v_s$. The only other columns of $p \to U$ which differ from U are the sth and the tth. If $s < t \leqslant m$, then $p \to U$ is the tableau

u_1		u_s		p		
v_1		u_t		v_t		
v_s						

where $u_s < u_t$ because $s < t$ and $p \leqslant v_t$. If $t > m$ there is nothing below p and $u_s < u_t$ because $s \leqslant m < t$. Finally, if $s = t$ then p is above u_t and $p \leqslant u_t$ so that $p \to U$ is standard.

It only remains to consider the case where u_t bumps no element in the second row, i.e., $u_t > v_s$. But this can only happen if $t > m$ and so $p \to U$ is the tableau

u_1		u_m	u_{m+1}		p		u_r
v_1		v_m	u_t				

which is standard because $u_{m+1} \leqslant u_t$.

We next describe a procedure reverse to the Schensted process. Let U^1 be a standard tableau in $\mathrm{Tab}_{\lambda^1}(S)$, and (i,j) the coordinates of an extremal box of the diagram of λ^1. (A box is extremal if it is an outside corner, i.e., if it has no neighbor to its right or below it. In terms of the coordinates (i,j) this means that $j = \lambda_i^1$ and $\lambda_i^1 > \lambda_{i+1}^1$.) Let $r = U^1(i,j)$ be the entry in this box. The reverse procedure is defined in the following recursive manner:

(1) Given that r is in the ith row of U^1 and U^1 is standard, there exists a biggest r_1 in the $(i-1)$st row such that $r_1 \leqslant r$. Replace r_1 by r.

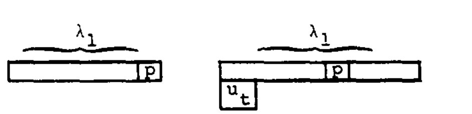
(2) Repeat step (1) with r_1 and the $(i-2)$nd row, and continue.

Proceeding in this way we obtain a new tableau U, which we denote by $U^1(i, \lambda_i) \leftarrow U^1$, of shape λ which is obtained from λ^1 by removing the extremal box (i, λ_i). We also end up with an element of S, namely, the element in the first row of U^1 which was bumped out in the final step of our recursive process. We call this the "bumped out" element of U^1.

The following lemma shows us that the procedure just described is inverse to the Schensted process.

LEMMA IV.1.2. *Let U be a standard tableau in* $\mathrm{Tab}_\lambda(S)$ *and $p \in S$. Let r be the entry in the extremal box $(i, \lambda_i + 1)$ of the tableau $U^1 = (p \to U)$ that was adjoined through the bumping of p into U, i.e., $r = U^1(i, \lambda_i + 1)$. Then $U = r \leftarrow U^1$ and the bumped out element of U^1 is p.*

Proof. We proceed by induction on the number of rows of λ. If λ has only one row, then $U^1 = (p \to U)$ is one of the following:



In the first case p is the extremal entry and the reverse process returns us to the original tableau U with p bumped out. In the second case, p has bumped u_t, and u_t is the adjoined extremal element of $p \to U$. Since $u_{t-1} < p \leqslant u_t <$

u_{t+1}, u_t bumps p in the reverse process. So we again return to U with p bumped out.

We now suppose λ has more than one row. The first step in the Schensted process leads to one of two situations: either p tacks on to the first row or it bumps some u_t in the first row. In the first case we have completed the process $p \to U$, and p is the adjoined extremal element. The reverse process then bumps out p and we return to U. In the second case, the remaining steps of the Schensted process can be described as $(u_t \to \tilde{U})^+$ where $\tilde{U}$ is the tableau U with the top row removed, and $(u_t \to \tilde{U})^+$ is the tableau obtained by adjoining the top row of $(p \to U)$ to $(u_t \to \tilde{U})$. We see that the adjoined extremal element r of $(p \to U)$ is the same as the adjoined extremal element of $(u_t \to \tilde{U})$. By induction, $\tilde{U} = r \leftarrow (u_t \to \tilde{U})$ and u_t is the bumped out element. Therefore in the last step of the reverse process $r \leftarrow U^1$, u_t bumps p and gives us U with p bumped out.

DEFINITION IV.1.3. A finite sequence $\mathbf{a} = (a_1,..., a_n)$ of positive integers is a word of Yamanouchi, or a Y-word, if for each $k = 1,..., n$, the number of times i appears in the sequence $(a_1,..., a_k)$ is not smaller than the number of times $i+1$ appears, for every positive integer i. If $\mathbf{a} = (a_1,..., a_n)$ is any sequence of positive integers, its content is defined to be the sequence $\lambda = (\lambda_1, \lambda_2,...)$, where $\lambda_i =$ the number of times i appears in $\mathbf{a}$. It follows that $\mathbf{a}$ is a word of Yamanouchi if and only if the content of $(a_1,..., a_k)$ is a partition for each $k = 1,..., n$. A finite sequence $(x_{i_1},..., x_{i_n})$ from an ordered set $S = \{x_1 \leqslant \cdots \leqslant x_m\}$ is called a Y-word if $(i_1,..., i_n)$ forms a Y-word of numbers. Words of Yamanouchi are also called lattice permutations in the literature.

There is a well-known bijection between the set of words of Yamanouchi of content λ and the set of standard tableaux of shape λ with distinct entries from the set $\{1,..., n\}$, where n is the weight of λ. To see this let T be a row-standard tableau of shape λ with distinct entries $1,..., n$. Each $i \in \{1,..., n\}$ appears as an entry $U(j, k)$. Let $a_i = j$ and define $\alpha(T)$ to be the sequence $(a_1,..., a_n)$ of content λ. Conversely, if $\mathbf{a} = (a_1,..., a_n)$ is a sequence of content λ, let $(a_{k_1},..., a_{k_{\lambda_i}})$ be the subsequence of $\mathbf{a}$ with entries equal to i. Define $\beta(\mathbf{a})$ to be the row-standard tableau of shape λ given by $\beta(\mathbf{a})(i,j) = k_j$. It is clear that α and β are inverses. An easy induction argument on n shows that β takes Y-words to standard tableaux and that α does the reverse, giving the desired bijection. As an illustration, β takes the Y-word $(1, 1, 2, 1, 3, 2)$ to the tableau

1	2	4
3	6	
5		

.

DEFINITION IV.1.4. Recall that if T is a tableau of shape λ, its transpose $\tilde{T}$ is the tableau of shape $\tilde{\lambda}$ given by $\tilde{T}(i,j) = T(j, i)$. It is clear that $\tilde{\tilde{T}} = T$

and that if T is a standard tableau with distinct entries, then so is $\tilde{T}$. Now let $\mathbf{a}=(a_1,..., a_n)$ be a word of Yamanouchi of content λ. We define its transpose $\tilde{\mathbf{a}}$ to be the Y-word $\alpha((\beta(\mathbf{a}))^{\sim})$ of content $\tilde{\lambda}$, so that $\tilde{\tilde{\mathbf{a}}}=\mathbf{a}$. It is easy to see that $\tilde{\mathbf{a}}=(\tilde{a}_1,..., \tilde{a}_n)$, where $\tilde{a}_i$ is the number of a_k such that $a_k=a_i$ and $k \leqslant i$.

Let λ and μ be partitions. We are going to define two sets A and B associated to the pair (λ, μ) and construct a bijection $\Phi: A \to B$. The injectivity of Φ will be proved in this section and the surjectivity in the next.

The set A is the set of pairs (U_1, U_2) where U_1 is a standard tableau in $\mathrm{Tab}_\lambda(S)$ and U_2 is a standard tableau in $\mathrm{Tab}_\mu(S)$.

The set B is the set of triples (ν, U, V) where ν is a partition containing λ, U is a standard tableau in $\mathrm{Tab}_\nu(S)$, and V is a standard tableau in $\mathrm{Tab}_{\nu/\lambda}(S)$ of content $\tilde{\mu}$ which satisfies the following condition: the sequence of entries of V obtained by reading each column of V from the bottom up, starting with the left-most column and moving to the right column by column, is a Y-word. More explicitly, if the diagram of ν/λ is $\{(i,j) \mid i=1,..., q,\ \lambda_i<j \leqslant \nu_i\}$ then the sequence $(V(q, \lambda_q+1), V(q-1, \lambda_q+1),..., V(q, \lambda_q+2),..., V(1, \lambda_1))$ must be a word of Yamanouchi.

As an illustration consider the following standard tableau of shape $(6, 4, 3, 3, 2)/(3, 2, 1)$ and content $(5, 5, 2)$:

			1	2	3
		1	2		
	1	2			
1	2	3			
1	2				

The associated sequence is $(1, 1, 2, 2, 1, 3, 2, 1, 2, 1, 2, 3)$ which is a Y-word.

We are now ready to define the map $\phi: A \to B$. Let $(U_1, U_2) \in A$. We bump the entries of U_2 successively into U_1 in the following recursive manner. We start with the first column (i.e., left-most column) of U_2 and label its entries, $p_1,..., p_{\tilde{\mu}_1}$, starting from the bottom and working up, so that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_{\tilde{\mu}_1}$. Then we perform the successive bumpings

$$(p_{\tilde{\mu}_1} \to (\cdots \to (p_2 \to (p_1 \to U_1)) \cdots)).$$

In this way we obtain a standard tableau U_1' and we let U_2' denote the tableau obtained from U_2 by removing its first column. We repeat the above procedure with U_1' and U_2'. Continuing in this manner we finally obtain a standard tableau U of some shape ν, where ν is a partition containing λ with $|\nu|-|\lambda|=|\mu|$. We construct a tableau $V \in \mathrm{Tab}_{\nu/\lambda}(S)$ as follows. Each box (l, k) in the diagram of ν/λ was adjoined as a result of having bumped some entry $U_2(i,j)$ of U_2 into U_1. We define $V(l, k)=j$. It is clear from the construction that the content of V is $\tilde{\mu}$.

In order to see that the triple (ν, U, V) is in B, it only remains to show that V is a standard tableau and that the sequence associated to V is a word of Yamanouchi.

To prove the standardness of V, we must simply show that if we start with a standard tableau U' of λ', then the shape of the tableau obtained from bumping a standard column (from bottom to top) into U' never has two boxes in the same row. More precisely:

LEMMA IV.1.5. *Let U' be a standard tableau of shape λ' and let $p_1 \geqslant \cdots \geqslant p_r$ be elements of S. Let U'' be the standard tableau*

$$p_r \to (\cdots \to (p_2 \to (p_1 \to U')) \cdots)$$

of some shape λ''. Then the diagram of λ''/λ' contains at most one box in each row. Moreover, the box adjoined by bumping in p_{i+1} is below the box adjoined by bumping in p_i, for $i = 1,\ldots, r-1$.

Proof. It is clearly enough to consider the case $r = 2$. If p does not bump any entry in the first row of U', then $p_1 \to U'$ has p_1 in the first row. Since $p_2 \leqslant p_1$, p_2 must bump some element in the first row of $p_1 \to U'$ so that $p_2 \to (p_1 \to U')$ has $\lambda_1 + 1$ boxes in the first row. If p_1 does bump some entry, p_1', from the first row, then p_2 must also bump some entry, p_2', with $p_2' \leqslant p_1'$, from the first row of $p_1 \to U'$. An induction on the number of rows of λ' (bumping p_1', p_2' successively into the lower rows of U') finishes the argument.

Next we prove that the sequence associated to V is a Y-word. We observe that the Yamanouchi condition depends only on the relative positions of the entries i and $i+1$ in V (for each i). Therefore it is sufficient to consider the case where μ has only two columns and to prove the following lemma.

LEMMA IV.1.6. *Let U_1 be a standard tableau of shape λ, U_2 be the standard tableau*

$$\begin{array}{|c|c|} \hline k_1 & \ell_1 \\ \hline \vdots & \vdots \\ \hline k_s & \ell_s \\ \hline \vdots \\ \cline{1-1} k_t \\ \cline{1-1} \end{array}$$

of shape μ, and U be the tableau

$$l_1 \to (\cdots (l_s \to (k_1 \to (\cdots \to (k_t \to U_1) \cdots))) \cdots)$$

of some shape λ'. Then the box of λ' adjoined to λ as a result of bumping in l_a is strictly to the right of the box of λ' adjoined to λ by bumping in k_a, for $a = 1,..., s$.

Proof. We may assume $s = t$ because we can replace U_1 by the tableau $k_{s+1} \to (\cdots \to (k_t \to U_1) \cdots)$ and remove $k_{s+1},..., k_t$ from U_2. Now we proceed by induction on s. The case $s = 1$ is clear because $l_1 > k_1$ implies that the elements displaced from each row as a result of bumping k_1 into U_1 are exceeded, row by row, by the elements displaced by the bumping of l_1 into $k_1 \to U_1$. Now let $s > 1$. Observe that

$$\begin{aligned} &l_s \to (k_1 \to (\cdots (k_s \to U_1) \cdots)) \\ &\quad = k_1 \to (\cdots \to (k_{s-1} \to (l_s \to (k_s \to U_1))) \cdots). \end{aligned} \tag{$*$}$$

Clearly it suffices to check $(*)$ when $s = 2$, i.e., we have $k_1 \leqslant k_2 < l_2$ and we want to show that $l_2 \to (k_1 \to (k_2 \to U_1)) = k_1 \to (l_2 \to (k_2 \to U_1))$. If l_2 exceeds every entry in the first row of U_1, then the result is clear. If l_2 bumps an element, say l_2', in the first row, then k_2 must have bumped an element k_2', and k_1 an element k_1' in the first row, with $k_1' \leqslant k_2' < l_2'$. An induction on the number of rows of U_1 finishes the proof of $(*)$. Using $(*)$, if we let $U_1' = l_s \to (k_s \to U)$, then we have $U = l_1 \to (\cdots \to (l_{s-1} \to (k_1 \to (\cdots \to (k_{s-1} \to U_1') \cdots))) \cdots)$ and we are done by the case $s = 1$ and induction on s.

PROPOSITION IV.1.7. *The map $\Phi: A \to B$ is an injection.*

Proof. Let $\bar{A} =$ the set of ordered pairs (U_1, U_2) with $U_1 \in \text{Tab}_\lambda(S)$ and $U_2 \in \text{Tab}_\mu(S)$. We will define a map $\Psi: B \to \bar{A}$ such that $\Psi \circ \Phi$ is the identity when restricted to A.

We start with $(\nu, U, V) \in B$. In the Y-word associated to V, the largest entry, μ_1, appears $\tilde{\mu}_t$ times where $\tilde{\mu} = (\tilde{\mu}_1,..., \tilde{\mu}_t)$ is the content of V. The first appearance of μ_1 corresponds to an extremal box of V. Since ν/λ is the shape of V, this box is also extremal for U, which is of shape ν. We can therefore perform the reverse Schensted process on U with respect to this extremal box, obtaining a new standard tableau whose shape is ν with that extremal box removed. After $\tilde{\mu}_t$ iterations of this procedure we obtain a standard tableau U' of shape ν' and $\tilde{\mu}_t$ elements which have been bumped out of U. We arrange these $\tilde{\mu}_t$ elements in a column, γ_t, in the order in which they were bumped out (i.e., the first one at the top, etc.).

Let V' be the standard tableau of shape ν'/λ obtained from V by removing all boxes of V with the entry μ_1. The sequence associated to V' is the subsequence of the Y-word associated to V obtained by deleting the elements μ_1, and is therefore also a Y-word. The triple (ν', U', V') is an element of the set B^1 associated to the pair of partitions λ, μ' where μ' is the partition whose transpose is $(\tilde{\mu}_1,..., \tilde{\mu}_{t-1})$. By induction on t (i.e., the number of columns of

μ), the map $\Psi': B' \to \bar{A}'$ associates to our triple (ν', U', V') a pair (U_1', U_2'), where $U_1' \in \text{Tab}_\lambda(S)$, $U_2' \in \text{Tab}_{\mu^1}(S)$. Define a tableau U_2 of shape μ by adjoining the column γ_t to the tableau U_2' as a last column. Let $U_1 = U_1'$ and define $\Psi(\nu, U, V) = (U_1, U_2) \in \bar{A}$.

Now that we have defined $\Psi: B \to \bar{A}$, we will show that if $(U_1, U_2) \in A$, then $\Psi(\Phi(U_1, U_2)) = (U_1, U_2)$, proceeding by an induction argument on the number of columns of U_2. Denote by U_2' the tableau obtained from U_2 by removing the last column. Let $(\nu', U', V') = \Phi'(U_1, U_2')$ and let $(\nu, U, V) = \Phi(U_1, U_2)$. From the definition of Φ and Φ', it is clear that U is the tableau obtained by bumping the last column of U_2 into U' and V' is obtained from V by removing all boxes having the largest entry μ_1. By the induction hypothesis we have that $\Psi'(\Phi'(U_1, U_2')) = (U_1, U_2')$. In order to finish the proof we have to show that the first $\tilde{\mu}_t$ steps in the definition of Ψ applied to (ν, U, V) give us the above U', V', and the last column of U_2 as the column γ_t of bumped out elements. But this is really the one column situation that starts our inductive proof, so we may assume that U_2 consists of a single column whose entries are $p_r \leqslant p_{r-1} \leqslant \cdots \leqslant p_1$.

Observe that $U = p_r \to (\cdots \to (p_1 \to U_1) \cdots)$ and that the box of ν (= shape of U) which is adjoined by the final bumping $p_r \to (\ )$ is the lowest (extremal) box of ν/λ (by Lemma IV.1.5). Thus, performing the inverse Schensted process on U with respect to this extremal box, which is the first step of Ψ, gives us back the tableau $p_{r-1} \to (\cdots \to (p_1 \to U_1) \cdots)$ with p_r bumped out (by Lemma IV.1.2). Induction on r now completes the proof.

IV.2. *The Proof of the Littlewood–Richardson Rule*

In this section we will prove the following theorem which is the Littlewood–Richardson rule formulated for Schur functors.

THEOREM IV.2.1. *Let R be a ring which contains a field K of characteristic zero and let λ, μ be partitions. For every finitely generated free R-module F, there is a natural isomorphism*

$$L_\lambda F \otimes L_\mu F \cong \sum_\nu (\lambda, \mu; \nu)\, L_\nu F$$

where $(\lambda, \mu; \nu)$ is the number of standard tableaux V in $\text{Tab}_{\nu/\lambda}(\{1, \ldots, |\mu|\})$ *of content $\tilde{\mu}$ such that the sequence associated to V (formed by listing the entries of V from bottom to top in each column, starting from the left-most column) is a word of Yamanouchi.*

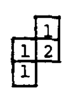
It is clear from the universality of Schur functors that we may assume $R = K$. Since K is a field of characteristic zero, the group $GL(n, K)$ is linearly reductive, and so the category of polynomial representations of $GL(n, k)$ is semisimple. Moreover, if F is a K-vector space, then the modules

$\{L_\nu(F) \mid \nu$ is a partition, $\nu_1 \leqslant n\}$ form a complete set of distinct irreducible polynomial representations of $GL(F)$. Therefore the tensor product $L_\lambda F \otimes L_\mu F$ breaks up into the direct sum $\sum_\nu a^\nu_{\lambda\mu} L_\nu F$ of irreducible representations and we have to show $a^\nu_{\lambda\mu} = (\lambda, \mu; \nu)$. We will proceed by constructing $(\lambda, \mu; \nu)$ morphisms $L_\nu F \to L_\lambda F \otimes L_\mu F$ and prove that their images are independent. This will give us an injection $\sum(\lambda, \mu; \nu) L_\nu F \to L_\lambda F \otimes L_\mu F$ which has to be an isomorphism because the dimension of $\sum(\lambda, \mu; \nu) L_\nu F$ as a K-vector space is $\geqslant$ the dimension of $L_\lambda F \otimes L_\mu F$. To see this inequality on the dimensions, let $S = \{x_1, ..., x_n\}$ be an ordered basis for F. Then the set A of Section IV.1 gives a K-basis of $L_\lambda F \otimes L_\mu F$ (by the Standard Basis Theorem for Schur functors—II.2.16) while the set B gives a basis of $\sum(\lambda, \mu; \nu) L_\nu F$ (again by II.2.16). By Proposition IV.1.7, the cardinality of A is $\leqslant$ the cardinality of B as desired.

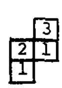
From the above discussion we see that in order to finish the proof, we need only construct $(\lambda, \mu; \nu)$ morphisms $L_\nu F \to L_\lambda F \otimes L_\mu F$ with independent images. Before we define these morphisms we need a combinatorial definition and lemma.

DEFINITION IV.2.2. Let V be a standard tableau of shape ν/λ and content $\tilde{\mu}$ such that the associated sequence $\mathbf{a} = (a_1, ..., a_m)$ of entries of V is a Y-word, where $m = |\mu|$.

Let $\tilde{\mathbf{a}} = (\tilde{a}_1, ..., \tilde{a}_m)$ be the transpose of $\mathbf{a}$ (IV.1.4). We construct a tableau $\hat{V}$ of shape ν/λ by replacing each entry a_i of V by $\tilde{a}_i$. Since the content of $\tilde{\mathbf{a}}$ is μ, the tableau $\hat{V}$ has content μ. As an example, if

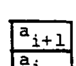
V = 

then

$\hat{V}$ =  .

LEMMA IV.2.3. *$\hat{V}$ is decreasing in rows and strictly decreasing in columns.*

Proof. To check the column condition, it is sufficient to consider what happens to two adjacent boxes



in the same column of V. Since V is standard, $a_{i+1} \leqslant a_i$. Recall that $\tilde{a}_i$ is the number of $a_k = a_i$ for $k \leqslant i$, and since $\mathbf{a}$ is a Y-word, we have $\tilde{a}_{i+1} > \tilde{a}_i$ as required. To see that $\hat{V}$ is decreasing in rows, consider two adjacent boxes $\boxed{a_i}\boxed{a_j}$ in the same row of V. If we remove from V all boxes containing entries $< a_i$, we still have a standard tableau whose associated sequence is a Y-word and the picture $\boxed{\tilde{a}_i}\boxed{\tilde{a}_j}$ in $\hat{V}$ is not changed. We may therefore assume that $a_i = 1$. Suppose first that there is nothing above the box $\boxed{a_i}$. Then $a_i < a_j$ implies that $\tilde{a}_i \geqslant \tilde{a}_j$ because $\mathbf{a}$ is a Y-word. If there is a box $\boxed{a_{i+1}}$ lying above the box $\boxed{a_i}$, then $a_{i+1} = a_i = 1$ by the standardness of V. Using induction on the number of boxes above $\boxed{a_i}$ we have $\tilde{a}_{i+1} \geqslant \tilde{a}_{j+1}$. We already know that $\tilde{a}_{j+1} > \tilde{a}_j$, so that $\tilde{a}_i = \tilde{a}_{i+1} - 1 \geqslant \tilde{a}_{j+1} - 1 \geqslant \tilde{a}_j$ as desired.

DEFINITION IV.2.4. Let V be as in IV.2.2. We define a map $\sigma_V : \Delta_{\nu/\lambda} \to \Delta_\mu$ from the diagram of ν/λ to the diagram of μ by $\sigma_V(i,j) = (\hat{V}(i,j), V(i,j))$. This definition can be reformulated as follows. For each $k = 1, ..., \tilde{\mu}_1$, there exist μ_k boxes (i,j) of ν/λ such that $\hat{V}(i,j) = k$. For a fixed k, consider the sequence $(i_1, j_1), ..., (i_{\mu_k}, j_{\mu_k})$ of such boxes formed by listing them from bottom to top in each column, starting from the left-most column. Notice that $i_1 \geqslant i_2 \geqslant \cdots$ and $j_1 < j_2 < \cdots$ by Lemma IV.2.3. Then the map σ_V takes (i_l, j_l) to (k, l).

If $\gamma \subseteq \nu$ are partitions we will denote by $\otimes_{\nu/\gamma}(F)$ the tensor product of $|\nu| - |\gamma|$ copies of F where each F is indexed by a pair (i,j) in the diagram of ν/γ. Given a tableau V as above, we define a map $\eta_V : \Lambda_{\nu/\lambda} F \to \Lambda_\mu F$ to be the composition $\Lambda_{\nu/\lambda} F \to^{\Delta} \otimes_{\nu/\lambda}(F) \to \otimes_\mu(F) \to^{m} \Lambda_\mu F$, where the middle map is the canonical isomorphism which sends the (i,j) copy of F to the $\sigma_V(i,j)$ copy of F by the identity 1_F. To describe η_V in terms of tableaux, if $T \in \mathrm{Tab}_{\nu/\lambda}(F)$, we let $T^V \in \mathrm{Tab}_\mu(F)$ denote the tableau given by $T^V(k,l) = T(\sigma_V^{-1}(k,l))$. Then $\eta_V(T) = \sum_\pi (-1)^\pi (T_\pi)^V$ where π runs over the row permutations of $\Delta_{\nu/\lambda}$ and $T_\pi \in \mathrm{Tab}_\lambda(F)$ is defined by $T_\pi(i,j) = T(\pi(i,j))$. Note that we are writing T in place of the element X_T in $\Lambda_{\nu/\lambda} F$ which corresponds to T.

Finally, we define a map $\phi_V : K_{\tilde{\nu}} F \to L_\lambda F \otimes L_\mu F$ to be the composition

$$K_{\tilde{\nu}} F \to \Lambda_\nu F \to \Lambda_\lambda F \otimes \Lambda_{\nu/\lambda} F \to \Lambda_\lambda F \otimes \Lambda_\mu F \to L_\lambda F \otimes L_\mu F$$

where the first map is the inclusion, the last map is $d_\lambda \otimes d_\mu$, the second map is the tensor product of the maps $\Delta : \Lambda^{\nu_i} F \to \Lambda^{\lambda_i} F \otimes \Lambda^{\nu_i - \lambda_i} F$, and the third map is $1 \otimes \eta_V$.

Since K is a field of characteristic zero, the $GL(F)$-modules $K_{\tilde{\nu}} F$ and $L_\nu F$ are irreducible and isomorphic, so we have our $(\lambda, \mu; \nu)$-morphisms $\phi_V : L_\nu F \to L_\lambda F \otimes L_\mu F$. All that remains is to show that the images of the ϕ_V are independent. For the remainder of this section fix an ordered basis $S = \{x_1, ..., x_n\}$ of F and let C_ν denote the canonical tableau in $\mathrm{Tab}_\nu(S)$, as in II.2.13 (i.e., $C_\nu(i,j) = x_j$). Consider the composite map $K_{\tilde{\nu}} F \to^{i} \Lambda_\nu F \to^{d_\nu} L_\nu F$.

It is easy to see that the map $d_{\tilde{v}}'\colon D_{\tilde{v}}F \to \Lambda_v F$ takes $X_{\tilde{C}_v} = (x_1)^{(\tilde{v}_1)} \otimes \cdots \otimes (x_t)^{(\tilde{v}_t)}$ to $X_{C_v} = x_1 \wedge \cdots \wedge x_{v_1} \otimes \cdots \otimes x_1 \wedge \cdots \wedge x_{v_q}$, so that $d_v \circ i$: $K_{\tilde{v}}F \to L_v F$ maps $d_{\tilde{v}}(X_{\tilde{C}_v})$ to $d_v(X_{C_v})$, which is non-zero because C_v is standard. Since the $GL(F)$-modules $K_{\tilde{v}}F$, L_vF are irreducible, $d_v \circ i$ must be an isomorphism. For the remainder of this section we will identify $K_{\tilde{v}}F$ and L_vF via the isomorphism $d_v \circ i$ and write C_v for the element $d_v(X_{C_v})$ in L_vF.

Keeping in mind the isomorphism $GL(F) \cong GL(n, K)$ determined by the fixed ordered basis S, the theorem from [8] quoted in the Appendix tells us that C_v is the unique $U^-(n, K)$-invariant element of L_vF, up to scalar multiple. We will use this observation in proving the final lemma in the proof of the Littlewood–Richardson rule.

LEMMA IV.2.5. *As V runs through the set of tableaux satisfying the conditions of* IV.2.2, *the morphisms ϕ_V give $(\lambda, \mu; v)$ independent copies of L_vF inside $L_\lambda F \otimes L_\mu F$.*

Proof. Since the element $\phi_V(C_v)$ is $U^-(n, K)$-invariant, it follows from the discussion preceding the lemma that it is sufficient to prove the set $\{\phi_V(C_v)\}$ to be linearly independent. Let τ be the K-linear map which projects $L_\lambda F$ onto the subspace (C_λ) spanned by C_λ. It is clearly sufficient to prove that $\{(\tau \otimes 1)(\phi_V(C_v))\}$ is a linearly independent subset of $(C_v) \otimes L_\mu F$. It is easy to see that $(\tau \otimes 1)(\phi_V(C_v)) = C_\lambda \otimes d_\mu(\eta_V(T))$, where $T \in \mathrm{Tab}_{v/\lambda}(S)$ is the tableau defined by $T(i, j) = x_j$. Therefore all we have to show is that $\{d_\mu(\eta_V(T))\}$ is a linearly independent subset of $L_\mu F$.

Recall from Definition IV.2.4 that $\eta_V(T) = \sum(-1)^\pi T_\pi^V$, where we are writing T_π^V for $(T_\pi)^V$. The linear independence of the set $\{d_\mu(\eta_V(T))\}$ will follow from the following two assertions:

(1) The tableaux $T^V \in \mathrm{Tab}_\mu(S)$ are standard (and distinct).

(2) When $d_\mu(\eta_V(T)) = \sum(-1)^\pi d_\mu(T_\pi^V)$ is expressed as a linear combination of standard tableaux, the coefficient of T_V is a positive integer.

First we prove that T^V is row-standard. Let us consider the kth row of T^V, where $1 \leqslant k \leqslant \tilde{\mu}_1$. Recall from Definition IV.2.4 that there is a sequence $(i_1, j_1), \ldots, (i_{\mu_k}, j_{\mu_k})$ of boxes of v/λ such that $\hat{V}(i_l, j_l) = k$ with $i_1 \geqslant i_2 \geqslant \cdots$ and $j_1 < j_2 < \cdots$. Since $\sigma_V(i_l, j_l) = (k, l)$, we have $T^V(k, l) = T(i_l, j_l) = j_l$, proving the standardness of the kth row. To prove column-standardness, we have to show $T^V(k, l) \leqslant T^V(k+1, l)$. Let (i_l, j_l), (i_l', j_l') be the boxes of v/λ such that $\sigma_V(i_l, j_l) = (k, l)$ and $\sigma_V(i_l', j_l') = (k+1, l)$. Then $T^V(k, l) = j_l'$ and $T^V(k+1, l) = j_l'$. In the enumeration of the Y-word of $\hat{V}$, (i_l, j_l) is the lth box in which k occurs and (i_l', j_l') is the lth box in which $k+1$ occurs. Since the lth time k appears must precede the lth time $k+1$ appears in the Y-word, we see that (i_l, j_l) must precede (i_l', j_l') in this enumeration. Therefore, $j_l \leqslant j_l'$ and (1) is proved.

To prove (2), we first observe that if π is a permutation leaving $\hat{V}$ invariant, i.e., $\hat{V}_\pi = \hat{V}$, then $(-1)^\pi d_\mu(T_\pi^V) = d_\mu(T^V)$. If $\hat{V}_\pi \neq \hat{V}$, then let k be the smallest entry of $\hat{V}$ which differs from the corresponding entry in $\hat{V}_\pi$. Since π permutes the rows of $\hat{V}$ and the rows of $\hat{V}$ are decreasing, there exists a box (i,j) of v/λ such that $\hat{V}(i,j) = k$ and $\hat{V}(\pi^{-1}(i,j)) > k$. Among all such boxes (i,j), choose the one with the largest first coordinate i and let $(i,j) = \pi(i,j')$. It is easy to see that in this case we have $(T_\pi^V)_{k,j'} > (T^V)_{k,j'}$ (see II.2.13). Therefore the coefficient of $d_\mu(T^V)$ in the expression of $d_\mu(T_\pi^V)$ as a linear combination of standard tableaux must be zero (by II.2.15). (2) now follows from this and the first observation, concluding the proof of the lemma and the proof of Theorem IV.2.1.

As a corollary of Theorem IV.2.1, we obtain analogs for Schur functors of the Pieri formulas for symmetric functions (or Schubert cycles).

COROLLARY IV.2.6. *If R contains a field of characteristic zero and F is a finitely generated free R-module, then there are natural isomorphisms*:

(a) $L_\lambda F \otimes S_p F \cong \sum_\nu L_\nu F$, *where ν runs over all partitions such that* $\lambda \subseteq \nu$, $|\nu| = |\lambda| + p$, *and* $\nu_i \leqslant \lambda_i + 1$ *for all* i.

(b) $L_\lambda F \otimes \Lambda^p F \cong \sum_\nu L_\nu F$, *where ν runs over all partitions such that* $\lambda \subseteq \nu$, $|\nu| = |\lambda| + p$, *and* $\tilde{\nu}_i \leqslant \tilde{\lambda}_i + 1$ *for all* i.

V. SCHUR COMPLEXES

V.1. *Definition, Universal Freeness, and Decomposition*

In this section F and G are finitely generated R-modules of ranks m and n, respectively, and $\phi: G \to F$ is an R-module homomorphism. We denote by c_ϕ the element of $F \otimes G^*$ corresponding to the map ϕ under the canonical isomorphism $\mathrm{Hom}_R(G, F) \cong F \otimes G^*$.

DEFINITION V.1.1. The symmetric algebra $S\phi$ of the morphism ϕ is the graded R-Hopf algebra $SF \otimes \Lambda G$ formed by taking the tensor product of the graded R-Hopf algebras SF and ΛG. We let $m_{S\phi}: S\phi \otimes S\phi \to S\phi$, $\Delta_{S\phi}: S\phi \to S\phi \otimes S\phi$, and $T_{S\phi}: S\phi \otimes S\phi \to S\phi \otimes S\phi$ denote the multiplication, comultiplication, and the twisting map of $S\phi$. We make $S\phi$ into a complex as follows: let $(S\phi)_j = \sum_{i=0}^{\infty} S_i F \otimes \Lambda^j G$ be the jth degree component of the complex and let the boundary map $\partial_{S\phi}$ be the collection $\{(\partial_{S\phi})_j\}$ where $(\partial_{S\phi})_j: (S\phi)_j \to (S\phi)_{j-1}$ is the R-map defined by the action $c_\phi \in SF \otimes \Lambda G^*$ on $SF \otimes \Lambda G$. Explicitly if $c_\phi = \sum f_t \otimes \gamma_t$ and $x \otimes y \in S_i F \otimes \Lambda^j G$ then $c_\phi(x \otimes y) = \sum f_t \cdot x \otimes \gamma_t(y)$. Equivalently, the action of c_ϕ on $S_i F \otimes \Lambda^j G$ can be thought of as the composition

$$S_iF \otimes \Lambda^j G \xrightarrow{1\otimes\Delta} S_iF \otimes G \otimes \Lambda^{j-1}G \xrightarrow{1\otimes\phi\otimes 1}$$

$$S_iF \otimes F \otimes \Lambda^{j-1}G \xrightarrow{m\otimes 1} S_{i+1}F \otimes \Lambda^{j-1}G.$$

LEMMA V.1.2. *$m_{S\phi}$, $\Delta_{S\phi}$, $T_{S\phi}$ are all compatible with the differential $\partial_{S\phi}$. (For example, $m_{S\phi}$ is compatible with $\partial_{S\phi}$ means that the diagram*

$$\begin{array}{ccc} S\phi \otimes S\phi & \xrightarrow{m_{S\phi}} & S\phi \\ \downarrow{\scriptstyle \partial_{S\phi\otimes S\phi}} & & \downarrow{\scriptstyle \partial_{S\phi}} \\ S\phi \otimes S\phi & \xrightarrow{m_{S\phi}} & S\phi \end{array}$$

is commutative where $\partial_{S\phi\otimes S\phi}$ is the differential of the tensor product of the complexes $S\phi$ and $S\phi$.)

DEFINITION V.1.3. $S_k\phi$ is the subcomplex of $S\phi$ given by

$$0 \to \Lambda^k G \to F \otimes \Lambda^{k-1}G \to \cdots \to S_{k-j}F \otimes \Lambda^j G \to \cdots \to S_kF \to 0$$

where the jth degree component $(S_k\phi)_j$ is $S_{k-j}F \otimes \Lambda^j G$.

Remarks. $S_0\phi$ is the complex $0 \to R \to 0$ where $(S_0\phi)_0 = R$. $S\phi$ is the direct sum $\sum_{k=0}^{\infty} S_k\phi$ of complexes. If ϕ is the map $0 \to F$ then $S_k\phi$ is the complex $0 \to S_kF \to 0$ where $(S_k\phi)_0 = S_kF$ and if ϕ is the map $G \to 0$ then $S_k\phi$ is the complex $0 \to \Lambda^k G \to 0$ where $(S_k\phi)_k = \Lambda^k G$.

DEFINITION V.1.4. The exterior algebra $\Lambda\phi$ of the morphism ϕ is the bigraded R-Hopf algebra $\Lambda F \dot{\oplus} DG$ formed by taking the antisymmetric tensor product of the graded R-Hopf algebras ΛF and DG (see Section I.6). We let $m_{\Lambda\phi}: \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi$, $\Delta_{\Lambda\phi}: \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$, and $T_{\Lambda\phi}: \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$ denote the multiplication, comultiplication, and the twisting map of $\Lambda\phi$. We make $\Lambda\phi$ into a complex as follows: let $(\Lambda\phi)_j = \sum_i \Lambda^i F \otimes D_j G$ be the jth degree component of the complex and let the boundary map $\partial_{\Lambda\phi}$ be the collection $\{(\partial_{\Lambda\phi})_j\}$ where $(\partial_{\Lambda\phi})_j: (\Lambda\phi)_j \to (\Lambda\phi)_{j-1}$ is the R-map defined by the action of $c_\phi \in \Lambda F \dot{\otimes} SG^*$ on $\Lambda F \dot{\otimes} DG$. Explicitly if $c_\phi = \sum f_t \otimes \gamma_t$ and $x \otimes y \in \Lambda^i F \otimes D_j G$ then $c_\phi(x \otimes y) = (-1)^i \sum f_t \wedge x \otimes \gamma_t(y)$.

LEMMA V.1.5. *$m_{\Lambda\phi}$, $\Delta_{\Lambda\phi}$, $T_{\Lambda\phi}$ are all compatible with the differential $\partial_{\Lambda\phi}$.*

DEFINITION V.1.6. $\Lambda^k\phi$ is the subcomplex of $\Lambda\phi$ given by

$$0 \to D_kG \to F \otimes D_{k-1}G \to \cdots \to \Lambda^{k-j}F \otimes D_jG \to \cdots \to \Lambda^k F \to 0$$

where the jth degree component $(\Lambda^k\phi)_j$ is $\Lambda^{k-j}F \otimes D_jG$.

Remarks. $\Lambda^0\phi$ is the complex $0 \to R \to 0$ where $(\Lambda^0\phi)_0 = R$. $\Lambda\phi$ is the direct sum $\sum_{k=0}^{\infty} \Lambda^k\phi$ of complexes. If ϕ is the map $0 \to F$ then $\Lambda^k\phi$ is the complex $0 \to \Lambda^k F \to 0$ where $(\Lambda^k F)_0 = \Lambda^k F$ and if ϕ is the map $G \to 0$ then $\Lambda^k\phi$ is the complex $0 \to D_k G \to 0$ where $(\Lambda^k\phi)_k = D_k G$.

Having at our disposal the complexes $\Lambda\phi$ and $S\phi$ corresponding to a map $\phi: G \to F$, we can proceed to construct complexes $L_{\lambda/\mu}\phi$ which are analogous to the Schur functors constructed in Chapter II. If $\mu \subseteq \lambda$ are partitions, we define complexes

$$\Lambda_{\lambda/\mu}\phi = \Lambda^{\lambda_1 - \mu_1}\phi \otimes \cdots \otimes \Lambda^{\lambda_q - \mu_q}\phi,$$

$$S_{\lambda/\mu}\phi = S_{\lambda_1 - \mu_1}\phi \otimes \cdots \otimes S_{\lambda_q - \mu_q}\phi.$$

Let a_{ij} be the $q \times t$ matrix $(t = \lambda_1)$ otained by setting

$$\begin{aligned} a_{ij} &= 1 \qquad \text{if} \quad \mu_i + 1 \leqslant j \leqslant \lambda_i \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

and define the map

$$d_{\lambda/\mu}\phi: \Lambda_{\lambda/\mu}\phi \to S_{\tilde{\lambda}/\tilde{\mu}}\phi$$

to be the composition

$$\begin{aligned} \Lambda_{\lambda/\mu}\phi &\to \Lambda^{a_{11}}\phi \otimes \cdots \otimes \Lambda^{a_{1t}}\phi \otimes \cdots \otimes \Lambda^{a_{q1}}\phi \otimes \cdots \otimes \Lambda^{a_{qt}}\phi \\ &\cong S_{a_{11}}\phi \otimes \cdots \otimes S_{a_{qt}}\phi \to S_{\tilde{\lambda}_1 - \tilde{\mu}_1}\phi \otimes \cdots \otimes S_{\tilde{\lambda}_t - \tilde{\mu}_t}\phi, \end{aligned}$$

where the first map is diagonalization, the middle map is the isomorphism identifying $\Lambda^{a_{ij}}\phi$ with $S_{a_{ij}}\phi$ (for $a_{ij} = 0$ or 1), and the last map is multiplication.

Since each of the maps comprising $d_{\lambda/\mu}\phi$ is a map of complexes, the map $d_{\lambda/\mu}\phi$ is itself a map of complexes. Hence its image is a complex and we make the following definition:

DEFINITION V.1.7. The image of $d_{\lambda/\mu}\phi$, denoted by $L_{\lambda/\mu}\phi$, is called the *Schur complex of ϕ of shape λ/μ.*

There are a few simple observations that we should make immediately:

(i) if $G = 0$, then $L_{\lambda/\mu}\phi = L_{\lambda/\mu}F$;

(ii) if $F = 0$, then $L_{\lambda/\mu}\phi \approx K_{\lambda/\mu}G$ in degree $|\lambda| - |\mu|$;

(iii) if $\lambda = (\lambda_1)$ and $\mu = 0$, then $L_{\lambda/\mu}\phi = \Lambda^{\lambda_1}\phi$;

(iv) if

$$\lambda = (\underbrace{1, \dots, 1}_{q})$$

and $\mu = 0$, then $L_{\lambda/\mu}\phi = S_q\phi$.

As with the Schur functors, we want to show that the Schur complexes $L_{\lambda/\mu}\phi$ are universally free. Again we shall resort to tableaux, but this time the situation is a bit more complex. Suppose we let $\{y_1, \dots, y_n\}$ be a basis for G, and $\{x_1, \dots, x_m\}$ a basis for F, and $S = \{y_1, \dots, y_n\} \cup \{x_1, \dots, x_m\}$. For simplicity, we shall totally order S by setting $y_i < x_j$ for all i, j, while maintaining the given orders among the x's and y's. Now, which tableaux in $\mathrm{Tab}_{\lambda/\mu}(S)$ would correspond in some canonical way to basis elements of $\Lambda_{\lambda/\mu}\phi$? If we demand that a tableau T in $\mathrm{Tab}_{\lambda/\mu}(S)$ be row-standard, we certainly could assign to each row, T^i, of T a basis element of $\Lambda^{\lambda_i - \mu_i}\phi$. For T^i would consist of the sequence $y_{i1}, \dots, y_{ik}$, $x_{i1}, \dots, x_{il}$ with $k + l = \lambda_i - \mu_i$, and to this sequence we would assign the element $y_{i1} \cdots y_{ik} \otimes x_{i1} \wedge \cdots \wedge x_{il} \in D_k G \otimes \Lambda^l F$, which would be a basis element of degree k in $\Lambda^{\lambda_i - \mu_i}$. However, the row-standardness of T would prevent us from realizing basis elements such as $y_1^{(k)} \otimes x_{i1} \wedge \cdots \wedge x_{il} \in D_k G \otimes \Lambda^l F$.

If, to remedy this, we restricted attention to tableaux which were co-row-standard, we obviously would mess up the basis elements for ΛF. This leads us to a modification of the definition of row- and co-row-standard. In fact, the modification we shall adopt does not require that the total order of S be the one discussed above. It will allow us, for example, to take any total order on S in which the prescribed orders on the bases of G and F are separately preserved.

DEFINITION V.1.8. Let S be a totally ordered set, let Y be a subset of S. A tableau $T \in \mathrm{Tab}_{\lambda/\mu}(S)$ is said to be *row-standard* mod Y if each row of T is non-decreasing, and if, when repeats occur in a row, they occur only among elements of Y. A tableau is *column-standard* mod Y if each column is non-decreasing, and if, when repeats occur in a column, they occur only among elements in the complement of Y. T is *standard* mod Y if T is row- and column-standard mod Y.

Notice that the previous definitions of row- (co-row-)standard, etc., are obtained by setting $Y = S$ or $Y = \varnothing$.

Returning to the situation of a map $\phi: G \to F$, with bases $Y = \{y_1, \dots, y_n\}$, $X = \{x_1, \dots, x_m\}$ of G and F, and $S = Y \cup X$, we will take any total order on S in which the given orders on Y and X are preserved. If, now, we take a tableau $T \in \mathrm{Tab}_{\lambda/\mu}(S)$ which is row-standard mod Y, then it is clear that to such a tableau we may associate a basis element of $\Lambda_{\lambda/\mu}\phi$. Conversely, given a basis element of $\Lambda_{\lambda/\mu}\phi$, we can assign to it a tableau in $\text{Tab}_{\lambda/\mu}(S)$ which is row-standard mod Y.

If we denote by Z_T the basis element of $\Lambda_{\lambda/\mu}\phi$ corresponding to the row-standard mod Y tableau T, then the elements $\{d_{\lambda/\mu}(Z_T)\}$ generate $L_{\lambda/\mu}\phi$. What we shall see is that the set $\{d_{\lambda/\mu}(Z_T)/T$ is standard mod $Y\}$ is a basis for $L_{\lambda/\mu}\phi$. The proof of this fact proceeds formally almost word for word as did the proofs of the universal freeness and standard basis theorems for $L_{\lambda/\mu}F$ and $K_{\lambda/\mu}F$ in II.2 and II.3. In fact, rather than give a formal proof here, we shall simply indicate the formal (and not so formal) analogs to the discussion in II.2 and II.3 (see [1] for details).

Items II.2.3 through II.2.6 are purely formal and carry over to our situation *mutatis mutandis*. In the proofs of II.2.7 and II.2.8, a reference to [3] was given but it was also shown (in II.2.8) that this reference could be replaced by the observation that the composition $\Lambda^2F \to F \otimes F \to S_2F$ is zero. Making the corresponding observation for $\Lambda^2\phi \to \Lambda^1\phi \otimes \Lambda^1\phi \to S_2\phi$, we have the analogs of II.2.7 and II.2.8. Slightly more complicated computation (or more elaborate decomposition of the maps involved and use of (co-)associativity and (co-)commutativity) gives us II.2.9 through II.2.11, so that we have, in particular, the map $\square_{\lambda/\mu}\phi$ whose image is contained in the kernel of $d_{\lambda/\mu}\phi$. As in II.2.12 we now make the definition:

DEFINITION V.1.9. For partitions $\mu \subseteq \lambda$, and a map $\phi: G \to F$, define $\bar{L}_{\lambda/\mu}\phi$ to be the cokernel of the map $\square_{\lambda/\mu}\phi$.

From this, we obtain the canonical surjection $\theta_{\lambda/\mu}: \bar{L}_{\lambda/\mu}\phi \to L_{\lambda/\mu}\phi$. The lemmas leading up to the proof of Theorem II.2.16, especially their versions in II.3 when these had to be given separate treatment (as in Lemma III.3.15), all carry over straightforwardly to give us:

THEOREM V.1.10 (The Standard Basis Theorem for Schur Complexes). *Let $\lambda = (\lambda_1,..., \lambda_q)$, $\mu = (\mu_1,..., \mu_q)$ be partitions with $\mu \subset \lambda$, and let $\phi: G \to F$ be a map of free modules. Let $Y = \{y_1,..., y_n\}$, $X = \{x_1,..., x_m\}$ be bases for G and F, and let $S = Y \cup X$ be totally ordered so that the orders of X and Y are preserved. Then $\{d_{\lambda/\mu}(Z_T)/T$ is a standard tableau* mod Y *in* $\text{Tab}_{\lambda/\mu}(S)\}$ *is a free basis for $L_{\lambda/\mu}\phi$, and the map $\theta_{\lambda/\mu}: \bar{L}_{\lambda/\mu}\phi \to L_{\lambda/\mu}\phi$ is an isomorphism. Hence $L_{\lambda/\mu}\phi$ is universally free (i.e., is a complex of universally free modules).*

(Recall that in our notation the element Z_T is the basis element of $\Lambda_{\lambda/\mu}\phi$ corresponding to the tableau T.)

The next step is to consider the analog of Theorem II.4.11, i.e., a decomposition theorem. Suppose, then, that $\phi_1: G_1 \to F_1$ and $\phi_2: G_2 \to F_2$ are maps of free modules, and let $\phi: G \to F$ be the direct sum of ϕ_1 and ϕ_2, with

$G = G_1 \oplus G_2$ and $F = F_1 \oplus F_2$. As in II.4.7 we make the following definition:

DEFINITION V.1.11. Let $\phi = \phi_1 \oplus \phi_2$ as above and let $\mu \subseteq \lambda$ be given partitions. If γ is any partition such that $\mu \subseteq \gamma \subseteq \lambda$, define subcomplexes $M_\gamma(\Lambda_{\lambda/\mu}\phi)$, $\dot{M}_\gamma(\Lambda_{\lambda/\mu}\phi)$ of $\Lambda_{\lambda/\mu}\phi$ as follows:

(i) $M_\gamma(\Lambda_{\lambda/\mu}\phi) = \text{Image}(\sum_{\mu \subseteq \sigma \subseteq \lambda, \gamma \leqslant \sigma} \Lambda_{\sigma/\mu}\phi_1 \otimes \Lambda_{\lambda/\sigma}\phi_2 \to \Lambda_{\lambda/\mu}\phi)$;

(ii) $\dot{M}_\gamma(\Lambda_{\lambda/\mu}\phi) = \text{Image}(\sum_{\mu \subseteq \sigma \subseteq \lambda, \gamma < \sigma} \Lambda_{\sigma/\mu}\phi_1 \otimes \Lambda_{\lambda/\sigma}\phi_2 \to \Lambda_{\lambda/\mu}\phi)$;

(iii) $M_\gamma(L_{\lambda/\mu}\phi) = d_{\lambda/\mu}(M_\gamma(\Lambda_{\lambda/\mu}\phi))$;

(iv) $\dot{M}_\gamma(L_{\lambda/\mu}\phi) = d_{\lambda/\mu}(\dot{M}_\gamma(\Lambda_{\lambda/\mu}\phi))$.

The proof of II.4.9 carries over to this more general situation to give:

PROPOSITION V.1.12. *The map*

$$\Lambda_{\gamma/\mu}\phi_1 \otimes \Lambda_{\lambda/\gamma}\phi_2 \to M_\gamma(\Lambda_{\lambda/\mu}\phi)$$

induces a map

$$L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2 \to M_\gamma(L_{\lambda/\mu}\phi)/\dot{M}_\gamma(L_{\lambda/\mu}\phi).$$

In order to make sense of this proof, we have to indicate what set of tableaux we are dealing with, as the tableaux are an essential ingredient of the proof of II.4.9. What we do is take ordered bases Y_1, Y_2 of G_1 and G_2, and X_1, X_2 of F_1 and F_2. We then take $S_1 = Y_1 \cup X_1$, $S_2 = Y_2 \cup X_2$, and $S = S_1 \cup S_2$ with the following order relations.

In S_i, every element of Y_i precedes every element of X_i for $i = 1, 2$, while the given orders in Y_i and X_i are preserved. In S, every element of S_1 precedes every element of S_2, and we let $Y = Y_1 \cup Y_2$. The tableaux we consider are elements of $\text{Tab}_{\lambda/\mu}(S)$, and row- (column-)standardness refers to standardness mod Y. [Observe that in this ordering of S, the basis $Y = Y_1 \cup Y_2$ of $G = G_1 \oplus G_2$ is *not* an initial segment of S, but we still have the fact that the standard tableaux mod Y yield a basis for $L_{\lambda/\mu}\phi$.] If $T \in \text{Tab}_{\lambda/\mu}(S)$, we define $\eta(T) \in \mathbb{N}^\infty$ to be the sequence: $\eta(T)_i = \mu_i +$ the number of elements of S_1 in the ith row of T. Now, with the usual preordering of $\text{Tab}_{\lambda/\mu}(S)$, the proof of Proposition V.1.12 proceeds as does the proof of II.4.9.

Finally, using the Standard Basis Theorem (Theorem V.1.10), we obtain:

THEOREM V.1.13. *The map*

$$L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2 \to M_\gamma(L_{\lambda/\mu}\phi)/\dot{M}_\gamma(L_{\lambda/\mu}\phi)$$

is an isomorphism. Hence, the complexes $\{M_\gamma(L_{\lambda/\mu}\phi)/\mu \subseteq \gamma \subseteq \lambda\}$ *give a* *filtration of the complex $L_{\lambda/\mu}\phi$ whose associated graded complex is isomorphic to $\sum_{\mu\subseteq\gamma\subseteq\lambda} L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2$.*

COROLLARY V.1.14. *Let $\phi: G\to F$ be any map of free modules, and let $\mu\subset\lambda$ be partitions. Denote by $(L_{\lambda/\mu}\phi)_j$ the component in degree j of the complex $L_{\lambda/\mu}\phi$. There is a natural filtration on $(L_{\lambda/\mu}\phi)_j$ whose associated graded module is*

$$\sum_{\substack{\mu\subseteq\gamma\subseteq\lambda\\ |\lambda|-|\gamma|=j}} L_{\gamma/\mu}F\otimes K_{\lambda/\gamma}G.$$

Proof. Observe that the module $(L_{\lambda/\mu}\phi)_j$ depends only on the modules F and G, and not on the map ϕ. Therefore, if we take $\phi_1: 0\to F$ and $\phi_2: G\to 0$, we have $(L_{\lambda/\mu}\phi)_j = (L_{\lambda/\mu}(\phi_1\oplus\phi_2))_j$ as R-modules. By our decomposition theorem, there is a natural filtration of $(L_{\lambda/\mu}(\phi_1\oplus\phi_2))_j$ whose associated graded object is $\sum_{\mu\subseteq\gamma\subseteq\lambda}(L_{\gamma/\mu}\phi_1\otimes L_{\lambda/\gamma}\phi_2)_j$. Noting that $(L_{\gamma/\mu}\phi_1)_i = 0$ if $i\neq 0$ and $(L_{\gamma/\mu}\phi_1)_0 = L_{\gamma/\mu}F$, we have $\sum_{\mu\subseteq\gamma\subseteq\lambda}(L_{\gamma/\mu}\phi_1\otimes L_{\lambda/\gamma}\phi_2)_j = \sum_{\mu\subseteq\gamma\subseteq\lambda} L_{\gamma/\mu}F\otimes(L_{\lambda/\gamma}\phi_2)_j$. Since $(L_{\lambda/\gamma}\phi_2)_j = 0$ if $|\lambda|-|\gamma|\neq j$, and $(L_{\lambda/\gamma}\phi_2)_j = K_{\lambda/\gamma}\phi_2$ if $|\lambda|-|\gamma|=j$, we get the desired result. [The last two assertions are observations (i) and (ii) following Definition V.1.7.]

COROLLARY V.1.15. *Let $\phi: G\to F$ be a split injection. Then $L_{\lambda/\mu}\phi$ is acyclic and $H_0(L_{\lambda/\mu}\phi) = L_{\lambda/\mu}(\text{Coker }\phi)$.*

Proof. We proceed by induction on rank G, the case $G=0$ being trivial. If rank $G>0$, we can split $\phi: G\to F$ into a direct sum: $\phi_1\oplus\phi_2: R\oplus G'\to R\oplus F'$, where $\phi_1: R\to R$ is the identity 1_R, and $\phi_2: G'\to F'$ is a split injection. By the decomposition theorem, $L_{\lambda/\mu}\phi$ decomposes into $\sum_{\mu\subseteq\gamma\subseteq\lambda} L_{\gamma/\mu}\phi_1\otimes L_{\lambda/\gamma}\phi_2$ up to filtration. It is easy to check that $L_{\gamma/\mu}(1_R)$ is exact for $\gamma\neq\mu$. It follows that $L_{\lambda/\mu}\phi$ is homotopically equivalent to $L_{\mu/\mu}\phi_1\otimes L_{\lambda/\mu}\phi_2$. But $L_{\mu/\mu}\phi_1$ is the complex $0\to R\to 0$ with R in degree zero. Therefore $L_{\lambda/\mu}\phi$ is homotopically equivalent to $L_{\lambda/\mu}\phi_2$. By induction, we are done.

COROLLARY V.1.16. *If $\phi=\phi_1\oplus\phi_2$, where ϕ_1 is an isomorphism, then the complex $L_{\lambda/\mu}\phi$ is homotopically equivalent to $L_{\lambda/\mu}\phi_2$.*

Proof. By V.1.15 the complex $L_{\gamma/\mu}\phi_1$ is exact if $\gamma\neq\mu$ and is the complex $0\to R\to 0$ if $\gamma=\mu$. The result follows from this observation and Theorem V.1.13.

Using the above results in conjunction with the *lemme d'acyclicité* (see [4, Lemma 4.1]), we can prove the acyclicity of a large family of Schur complexes.

THEOREM V.1.17. *Let* $\phi: G \to F$ *be a map of free* R*-modules where* $m = \dim F \geqslant n = \dim G$ *and let* $\lambda = (\lambda_1, \ldots, \lambda_p)$ *be a partition where* $\lambda_1 \leqslant m - n + 1$. *Suppose that for each* $j = 1, \ldots, n$ *the ideal* $I_j(\phi)$ *generated by the j-by-j minors of* ϕ *has grade* $\geqslant (m - n + 1)(n - j + 1)$. *Then the complex* $L_\lambda \phi$ *is acyclic.*

Proof. We proceed by induction on n, the case $n = 0$ being trivial. Suppose $n \geqslant 1$. By the Standard Basis Theorem for Schur Complexes (V.1.10) the length of $L_\lambda \phi$ is $\leqslant (m - n + 1)n$. By the acyclicity lemma it is sufficient to prove that $L_\lambda \phi$ is acyclic after localizing at primes P where $\operatorname{grade}(PR_P) < \operatorname{length} L_\lambda \phi$. But for such P $I_1(\phi_P) = R_P$ because $\operatorname{grade} I_1(\phi_P) \geqslant (m - n + 1)n > \operatorname{grade}(PR_P)$. This means some entry of the matrix ϕ_P must be a unit so that by a change of basis we can write the map $\phi_P: G_P \to F_P$ as the direct sum $\phi^1 \oplus 1: G^1 \oplus R_P \to F^1 \oplus R_P$, where F^1 and G^1 are free R_P-modules with $\dim F^1 = m - 1$, $\dim G^1 = n - 1$. Observing that for $j = 1, \ldots, n - 1$, $I_j(\phi^1) = I_{j+1}(\phi_P)$ we see that $\operatorname{grade} I_j(\phi^1) = \operatorname{grade} I_{j+1}(\phi_P) \geqslant (m - n + 1)(n - (j + 1) + 1) = ((m - 1) - (n - 1) + 1)(n - 1 - j + 1)$. Therefore, the map $\phi^1: G^1 \to F^1$ satisfies the required grade conditions and so $L_\lambda \phi^1$ is acyclic by induction. But now we are done because $L_\lambda \phi_P$ and $L_\lambda \phi^1$ are homotopically equivalent by (V.1.16).

One should note that if the map ϕ is generic, then $\operatorname{grade} I_j(\phi) = (m - j + 1)(n - j + 1) \geqslant (m - n + 1)(n - j + 1)$ so that the above theorem applies.

V.2. *Schur Functors of General Modules; Applications of Schur Complexes to Resolutions*

If M is an arbitrary R-module, its exterior algebra $\Lambda_R M = \sum_{i=0}^{\infty} \Lambda^i M$ is defined as the quotient of the tensor algebra $T_R M = \sum_{i=0}^{\infty} T_i M$ by the homogeneous two-sided ideal generated by the elements $\{m \otimes m \mid m \in M\}$. $\Lambda(-)$ is a right-exact additive functor from the category of R-modules to the category of Hopf algebras over R. The right exactness of $\Lambda(-)$ is a consequence of the following property: if N is an R-submodule of M and $\pi: M \to M/N$ is the canonical projection, then the kernel of the epimorphism $\Lambda(\pi): \Lambda(M) \to \Lambda(M/N)$ is the two-sided ideal of $\Lambda(M)$ generated by N.

Suppose now that the module M has a finite presentation, i.e., there is an exact sequence $G \to F \to M \to 0$ of R-modules where G and F are finitely generated and free. It follows from the above discussion that $\Lambda^p M$ has a finite presentation:

$$\Lambda^{p-1} F \otimes G \to \Lambda^p F \to \Lambda^p M \to 0.$$

We use these observations to motivate the definitions of the Schur functor $L_{\lambda/\mu} M$ for an arbitrary R-module M and discuss analogous presentations of them. For convenience we introduce the following notation. Let

$\mu = (\mu_1, ..., \mu_q) \subseteq \lambda = (\lambda_1, ..., \lambda_q)$ be partitions and let $p_i = \lambda_i - \mu_i$. Recall that $\Lambda_{\lambda/\mu}$ denotes the functor $\Lambda^{p_1} \otimes \cdots \otimes \Lambda^{p_q}$. We define the functor $\Lambda_{(\lambda/\mu)+}$ to be the direct sum

$$\sum_{i=1}^{q-1} \sum_{t=\mu_i-\mu_{i+1}}^{\lambda_{i+1}-\mu_{i+1}} \Lambda^{p_1} \otimes \cdots \otimes \Lambda^{p_{i-1}} \otimes \Lambda^{p_i+t} \otimes \Lambda^{p_{i+1}-t} \otimes \Lambda^{p_{i+2}} \otimes \cdots \otimes \Lambda^{p_q}.$$

If $q < 2$ we set $\Lambda_{(\lambda/\mu)+} = 0$. We define a map

$$\square_{\lambda/\mu} M: \Lambda_{(\lambda/\mu)+} M \rightarrow \Lambda_{\lambda/\mu} M$$

in a manner analogous to the definition of $\square_{\lambda/\mu} F$, where F is a free R-module. We start with the case $q = 2$, where $\square_{\lambda/\mu} M$ is defined to be

$$\sum_{t=\mu_1-\mu_2+1}^{\lambda_2-\mu_2} \Lambda^{p_1+t} M \otimes \Lambda^{p_2-t} M \xrightarrow{\square_{\Lambda M}} \Lambda^{p_1} M \otimes \Lambda^{p_2} M.$$

If $q > 2$ we set $\lambda^i = (\lambda_i, \lambda_{i+1})$, $\mu^i = (\mu_i, \mu_{i+1})$ and define $\square_{\lambda/\mu} M$, as in Definition II.2.10, to be the map

$$\sum_{i=1}^{q-1} 1_1 \otimes \cdots \otimes 1_{i-1} \otimes \square_{\lambda^i/\mu^i} M \otimes 1_{i+2} \otimes \cdots \otimes 1_q.$$

It is easy to see that $\square_{\lambda/\mu}(-)$ is a natural transformation of functors $\Lambda_{(\lambda/\mu)+}(-) \rightarrow \Lambda_{\lambda/\mu}(-)$.

DEFINITION V.2.1. $L_{\lambda/\mu} M$ is defined as the cokernel of the map $\square_{\lambda/\mu} M$. $L_{\lambda/\mu}(-)$ is a functor from R-modules to R-modules.

Because of Theorem II.2.16 the above definition agrees with the old definition of $L_{\lambda/\mu} F$ for free modules F. It is also clear from the definitions that if $\lambda = (q)$ then $L_\lambda M = \Lambda^q M$. If $\lambda = (\tilde{q}) = (1, ..., 1)$, then it is also true that $L_\lambda M = S_q M$. In order to see this recall that the symmetric algebra $S_R(M) = \sum_{i=0}^{\infty} S_i(M)$ is defined as the quotient of the tensor algebra $T_R M$ by the homogeneous two-sided ideal generated by the elements $\{m_1 \otimes m_2 - m_2 \otimes m_1 \mid m_1, m_2 \in M\}$, i.e., generated by the image of the map $\Delta: \Lambda^2 M \rightarrow M \otimes M = T_2 M$. Therefore, $S_q M$ is the cokernel of the map

$$\sum_{i=1}^{q-1} M \otimes \cdots \otimes \Lambda^2 M \otimes \cdots \otimes M \xrightarrow{1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1} M \otimes \cdots \otimes M$$

which is $L_{(\tilde{q})} M$ by definition.

Proposition V.2.2. *Suppose M has a finite presentation $G \to^{\phi} F \to M \to 0$. Then $L_\lambda M$ has a finite presentation*

$$L_{\lambda/1} F \otimes G \to L_\lambda F \to L_\lambda M \to 0.$$

More generally, $H_0(L_{\lambda/\mu}\phi) = L_{\lambda/\mu} M$.

Proof. From Corollary V.1.14 we know that $(L_\lambda(\phi))_0 = L_\lambda(F)$ and $(L_\lambda(\phi))_1 = L_{\lambda/1}(F) \otimes G$ so that the first assertion really is a special case of the second. The exactness of

$$\Lambda^{i-1} F \otimes G \to \Lambda^i F \to \Lambda^i M \to 0$$

tells us that $H_0(\Lambda^i \phi) = \Lambda^i M$. By a standard argument this implies $H_0(\Lambda^{i_1}\phi \otimes \cdots \otimes \Lambda^{i_q}\phi) = \Lambda^{i_1} M \otimes \cdots \otimes \Lambda^{i_q} M$. Now we have a commutative diagram

$$\begin{array}{ccccccc}
(\Lambda_{(\lambda/\mu)^+}(\phi))_1 & \longrightarrow & (\Lambda_{(\lambda/\mu)^+}(\phi))_0 & \longrightarrow & \Lambda_{(\lambda/\mu)^+}(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
(\Lambda_{\lambda/\mu}(\phi))_1 & \longrightarrow & (\Lambda_{\lambda/\mu}(\phi))_0 & \longrightarrow & \Lambda_{\lambda/\mu}(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
(L_{\lambda/\mu}(\phi))_1 & \longrightarrow & (L_{\lambda/\mu}(\phi))_0 & & & & \\
\downarrow & & \downarrow & & & & \\
0 & \longrightarrow & 0 & & & &
\end{array}$$

with exact rows and columns. A standard diagram chase gives the desired isomorphism from the cokernel of the third row, $H_0(L_{\lambda/\mu}(\phi))$, to the cokernel of the third column, $L_{\lambda/\mu}(M)$.

We now restrict our attention to the case when M is the cokernel of a generic map ϕ. For convenience we will write the map ϕ in the form $G \to F^*$, where G, F are free R-modules, $F^* = \mathrm{Hom}(F, R)$, so that we have an associated pairing $\langle\,,\rangle_\phi \colon F \otimes G \to R$. Suppose further that $m = \dim F \geqslant n = \dim G$. One way to obtain information about the module $L_\lambda M$ is through the use of an explicit minimal free resolution for $L_\lambda M$. A well-known example of such a resolution is the Eagon–Northcott complex

$$0 \to \Lambda^m F \otimes D_{m-n} G \to \cdots \to \Lambda^{n+i} F \otimes D_i G \to \cdots \to \Lambda^{n+1} F \otimes G \to \Lambda^n F \to 0$$

where the boundary maps are compositions of the type

$$\Lambda^j F \otimes D_i G \xrightarrow{\Delta \otimes \Delta} \Lambda^{j-1} F \otimes F \otimes D_{i-1} G \otimes G \to \Lambda^{j-1} F \otimes D_{i-1} G \otimes F \otimes G$$

$$\xrightarrow{1 \otimes \langle\,,\rangle_\phi} \Lambda^{j-1} F \otimes D_{i-1} G.$$

Observe that the above complex is isomorphic to the Schur complex $\Lambda^{m-n}(\phi)$ via the canonical isomorphisms $\Lambda^{n+i}F \otimes \Lambda^m F^* \cong \Lambda^{m-n-i}F^*$. Therefore it is a minimal free resolution of $H_0(\Lambda^{m-n}(\phi)) = \Lambda^{m-n}(M)$ which is isomorphic to the ideal $I_n(\phi)$ of maximal minors of the generic matrix ϕ.

Another example is provided by the similar complex

$$0 \to \Lambda^m F \otimes D_{m-n+1} G \to \cdots \to \Lambda^{n-i-1} F \otimes D_i G \to \cdots \to \Lambda^n F \otimes G \\ \to \Lambda^{n-1} F \to 0$$

which is isomorphic to $\Lambda^{m-n+1}(\phi)$ and is a resolution of $\Lambda^{m-n+1}(M)$. These two examples are special cases of the family of minimal free resolutions $L_\lambda(\phi)$ of the modules $L_\lambda(M)$ provided by Theorem V.1.17 and Proposition V.2.2 under the condition $\lambda_1 \leqslant m-n+1$.

Recall that when we say $\phi: G \to F^*$ is a generic map of free R-modules we mean that the entries of a matrix $X = (x_{ij})$ of ϕ are variables over a commutative ring K and that R is the polynomial ring $K[X]$ in the variables $\{x_{ij}\}$ over K. In order to emphasize the role of K we will write $\phi_K: G_K \to F_K^*$ and $M_K = \operatorname{Coker}(\phi_K)$. Note that M_K is a universal module in the sense that it commutes with change of the ring K, i.e., if $K_1 \to K_2$ is a ring homomorphism then $M_{K_1} \otimes_{R_1} R_2 = M_{K_2}$ as R_2-modules where $R_i = K_i[X]$. The following lemma, which we shall make use of in the sequel, illustrates the use of resolutions to obtain information about modules.

LEMMA V.2.3. *$\Lambda^{m-n+1}(M_K)$ is a free K-module.*

Proof. M_K is a graded module over the graded ring $K[X]$ and the complex $\mathbb{X}_K = \Lambda^{m-n+1}(\phi_K)$ is a graded minimal free resolution of the graded module $H_K = \Lambda^{m-n+1}(M_K)$ over the graded ring $K[X]$. Let $X_K(i) = \Lambda^{m-n+1-i}(F_K^*) \otimes D_i(G_K)$ be the ith degree chain module of the complex $\mathbb{X}_K$. Then $X_K(i) \cong \Lambda^{m-n+1-i}(F_0) \otimes_K D_i(G_0) \otimes_K K[X]$ as graded $K[X]$-modules where F_0, G_0 are free K-modules such that $F_K = F_0 \otimes_K K[X]$ and $G_K = G_0 \otimes K[X]$. Therefore the homogeneous components of $X_K(i)$ are finitely generated free K-modules and commute with change of the ring K. It follows that every homogeneous component H_K^t of H_K has a resolution

$$0 \to X_K^t(m-n+1) \to \cdots \to X_K^t(1) \to X_K^t(0) \to 0$$

by finitely generated free K-modules and this resolution commutes with change of the ring K. This means that if K is a field M_K^t is a vector space over K of dimension $= \sum_{i=0}^{m-n+1} (-1)^i \dim X_K^t(i)$ which is independent of the field K. Since $H_{\mathbb{Z}}^p \otimes K = H_K^p$ it follows that $H_{\mathbb{Z}}^t \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H_{\mathbb{Z}}^t \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})$ have the same dimension for every prime p. Consequently, $H_{\mathbb{Z}}^t$ must be a free $\mathbb{Z}$-module and therefore $H_K^t = H_{\mathbb{Z}}^t \otimes K$ is a free K-module.

We conclude the section by constructing a minimal free resolution of the module $\Lambda^{m-n+2}M$ as the mapping cone of a map between Schur complexes $L_{(m-n+1)\times 2}(\phi) \to \Lambda^{m-n+2}(\phi)$, where $(m-n+1)\times 2$ denotes the partition $(m-n+1, m-n+1)$. Recall that the complex $L_{(m-n+1)\times 2}(\phi)$ is acyclic and resolves the module $L_{(m-n+1)\times 2}(M)$. We will prove that $H_i(\Lambda^{m-n+2}(\phi))=0$ for $i \geqslant 2$ and that there is a natural map $L_{(m-n+1)\times 2}(M) \to^{\alpha_*} H_1(\Lambda^{m-n+2}(\phi))$. The map α_* can then be lifted, by the comparison theorem, to a map of Schur complexes as desired.

PROPOSITION V.2.4. (a) *$H_i(\Lambda^{m-n+2}(\phi))=0$ for $i \geqslant 2$.*

(b) *There is an exact sequence*

$$0 \to H_1(\Lambda^{m-n+2}\phi) \to \Lambda^{m-n+1}(M) \otimes G \xrightarrow{1\otimes\phi} \Lambda^{m-n+1}(M) \otimes F^*.$$

(c) *Define $\alpha: \Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G \to \Lambda^{n-1}F \otimes G$ to be the composition*

$$\Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G \xrightarrow{1\otimes 1\otimes \Delta} \Lambda^{n-1}F \otimes \Lambda^{n-1}F$$

$$\otimes \Lambda^{n-1}G \otimes G \xrightarrow{1\otimes\langle\,,\,\rangle_{\Lambda^{n-1}\phi}\otimes 1} \Lambda^{n-1}F \otimes G.$$

Then the image of the composition $\bar{\alpha} = (\pi \otimes 1)\circ \alpha$

$$\Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G \xrightarrow{\alpha} \Lambda^{n-1}F \otimes G \xrightarrow{\pi\otimes 1} \Lambda^{m-n+1}M \otimes G$$

is contained in $H_1(\Lambda^{m-n+2}(\phi))$.

(d) *The map $\bar{\alpha}$ induces a map $\alpha_* \cdot L_{(m-n+1)\times 2}(M) \to H_1(\Lambda^{m-n+2}(\phi))$.*

Proof. (a) Let $X = \{0 \to X_{m-n+2} \to \cdots \to X_1 \to X_0 \to 0\}$ be the Schur complex $\Lambda^{m-n+2}(\phi)$ and let $Y = \{0 \to Y_{m-n+1} \to \cdots \to Y_0 \to 0\}$ be the complex derived from X by taking $Y_i = X_{i+1}$ for $i \geqslant 0$ and the appropriate boundary maps. Clearly, $H_i(Y) = H_{i+1}(X)$ for $i \geqslant 1$. So we have to show that Y is acyclic. But $\operatorname{length}(Y) = m-n+1$ so by the acyclicity lemma it suffices to prove that Y is acyclic after we localize at primes P of $\operatorname{grade}(PR_P) < m-n+1$. But $\operatorname{grade} I_n(\phi_P) \geqslant I_n(\phi) = m-n+1$ because ϕ is generic, so that $I_n(\phi_P) = R_P$. Therefore ϕ_P is a split injection and by V.1.15 the complex $X_P = \Lambda^{m-n+2}(\phi_P)$ is acyclic, thus implying the acyclicity of Y_P.

(b) There is a short exact sequence of Schur complexes

$$0 \to \Lambda^{m-n+2}(\phi) \to \Lambda^{m-n+1}(\phi) \otimes \Lambda^1(\phi) \to L_{(m-n+1,1)}(\phi) \to 0. \qquad (*)$$

By V.1.15 $L_{(m-n+1,1)}(\phi)$ is an acyclic complex so that the long exact homology sequence of $(*)$ gives us $H_i(\Lambda^{m-n+2}\phi) \cong H_i(\Lambda^{m-n+1}\phi \otimes \Lambda^1\phi)$ for

$i \geqslant 1$. Again by V.1.15 and V.2 the complex $\Lambda^{m-n+1}\phi$ is acyclic and resolves the module $\Lambda^{m-n+1}M$. Now $\Lambda^1\phi$ is just the map ϕ viewed as a complex $(0 \to G \to^{\phi} F^* \to 0)$ and so $\Lambda^{m-n+1}\phi \otimes \Lambda^1\phi$ is the mapping cone of the map of complexes

$$1 \otimes \phi\colon \Lambda^{m-n+1}(\phi) \otimes G \to \Lambda^{m-n+1}(\phi) \otimes F^*.$$

From the associated long exact homology sequence we deduce an exact sequence

$$0 \to H_1(\Lambda^{m-n+1}\phi \otimes \Lambda^1\phi) \to \Lambda^{m-n+1}(M) \otimes G \to \Lambda^{m-n+1}(M) \otimes F^*$$

thus proving (b). We also get $H_i(\Lambda^{m-n+1}\phi \otimes \Lambda^1\phi) = 0$ for $i \geqslant 2$, giving another proof of (a).

(c) From V.2.2 and II.4.2 we have the exact sequences

$$\Lambda^{n-1}F \otimes G \xrightarrow{\partial_\phi} \Lambda^{n-2}F \to \Lambda^{m-n+2}M \to 0,$$

$$L_{n,n-1}F \otimes G \xrightarrow{\partial_\phi} L_{(n-1)\times 2}F \to L_{(m-n+1)\times 2}M \to 0,$$

where we use ∂_ϕ to denote the boundary map of any Schur complex associated to ϕ.

In order to show $\operatorname{Im}(\bar{\alpha}) \subseteq H_1(\Lambda^{m-n+2}(\phi))$, it suffices by part (b) to show that the composition

$$\Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G \xrightarrow{\bar{\alpha}} \Lambda^{m-n+1}M \otimes G \xrightarrow{1 \otimes \phi} \Lambda^{m-n+1}M \otimes F^*$$

is zero. But since $\bar{\alpha} = (\pi \otimes 1) \circ \alpha$ this is equal to the composition

$$\Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G \xrightarrow{\alpha} \Lambda^{n-1}F \otimes G \xrightarrow{\pi \otimes \phi} \Lambda^{m-n+1}M \otimes F^*.$$

Let $\{e_1,..., e_m\}$ and $\{\varepsilon_1,..., \varepsilon_m\}$ be dual bases for F and F^*, respectively, and let $\{g_1,..., g_n\}$ be a basis for G. To show $(\pi \otimes \phi) \circ \alpha = 0$ we start with $x \otimes y \otimes g_1 \wedge \cdots \wedge g_n \in \Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G$. Then $(\pi \otimes \phi)(\alpha(x \otimes y \otimes g_1 \wedge \cdots \wedge g_n)) = \pi \otimes \phi(\sum_{i=1}^n (-1)^i \langle y, g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge g_n \rangle x \otimes g_i) = \sum_{i=1}^n \sum_{j=1}^m (-1)^i \langle y, g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge g_n \rangle \langle e_j, g_n \rangle \pi(x) \otimes \varepsilon_j$ and this, by the Laplace expansion formula for determinants, is equal to $\sum_{j=1}^n \langle y \wedge e_j, g_1 \wedge \cdots \wedge g_n \rangle \pi(x) \otimes \varepsilon_j$. Now $\langle y \wedge e_j, g_1 \wedge \cdots \wedge g_n \rangle$ is in the ideal $I_n(\phi)$ of $n \times n$ minors of ϕ, $\pi(x) \in \Lambda^{m-n+1}M$, and $I_n(\phi)$ is the $(m-n+1)$st fitting ideal $F_{m-n+1}(M)$ of the module M. In general, $F_k(M)$ is contained in $\operatorname{Ann}(\Lambda^k M)$ for any module M of finite presentation (see [5]) so that $F_{m-n+1}(M)$ annihilates the module $\Lambda^{m-n+1}M$. Therefore, $\langle y \wedge e_j, g_1 \wedge \cdots \wedge g_n \rangle \pi(x) = 0$ for each j and this proves $(1 \otimes \phi) \circ \alpha = 0$ as desired.

(d) Recall that $L_{(m-n+1)\times 2}M$ is a quotient of $L_{(n-1)\times 2}F$ which is in turn a quotient of $\Lambda^{n-1}F \otimes \Lambda^{n-1}F$. We first show that $\bar{\alpha}$ induces a map $\bar{\alpha}'$ as indicated below

$$\begin{array}{ccc} \Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G & \xrightarrow{\psi \otimes 1} & L_{(n-1)\times 2}F \otimes \Lambda^n G \\ & \bar{\alpha} \searrow \quad \swarrow \bar{\alpha}' & \\ & \Lambda^{m-n+1}M \otimes G & \end{array}$$

where ψ is the natural projection. Since all the modules above are universal (commute with change of ring) it is sufficient to consider the case where $R = \mathbb{Z}[X]$ is the polynomial ring over $\mathbb{Z}$ in the entries of the generic matrix X defining the map ϕ. But all of these modules are free $\mathbb{Z}$-modules ($\Lambda^{m-n+1}M$ is $\mathbb{Z}$-free by (V.2.3) and the others are free over R). Therefore tensoring everything by $\mathbb{Q}$ we see that it is enough to consider the case $R = \mathbb{Q}[X]$. Now since we are over $\mathbb{Q}$ we know that $L_{(n-1)\times 2}$ is the cokernel of the map $\square: \Lambda^n F \otimes \Lambda^{n-2}F \to \Lambda^{n-1}F \otimes \Lambda^{n-1}F$. So we merely have to show that the composition

$$\Lambda^n F \otimes \Lambda^{n-2}F \otimes \Lambda^n G \xrightarrow{\square \otimes 1} \Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes G \xrightarrow{\alpha} \Lambda^{m-n+1}M \otimes G$$

is zero. There is a commutative diagram

$$\begin{array}{ccc} \Lambda^n F \otimes \Lambda^{n-2}F \otimes \Lambda^n G & \xrightarrow{\square \otimes 1} & \Lambda^{n-1}F \otimes \Lambda^{n-1}F \otimes \Lambda^n G \\ \downarrow \beta & & \downarrow \alpha \\ \Lambda^n F \otimes G \otimes G & \xrightarrow{\partial_\phi \otimes 1} & \Lambda^{n-1}F \otimes G \end{array}$$

where $\beta(x \otimes y \otimes g_1 \wedge \cdots \wedge g_n) = \sum_{i,j=1}^n (-1)^{i+j} \langle y, g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge \hat{g_j} \wedge \cdots \wedge g_n \rangle x \otimes g_i \otimes g_j$. Therefore $\bar{\alpha} \circ (\square \otimes 1) = (\pi \otimes 1) \circ \alpha \circ (\square \otimes 1) = (\pi \otimes 1) \circ (\partial_\phi \otimes 1) \circ \beta$ which is zero because the composition $\Lambda^n F \otimes G \to^{\partial_\phi} \Lambda^{n-1}F \to^{\pi} \Lambda^{m-n+1}M$ is zero. Checking the commutativity of the diagram we have

$$(\partial_\phi \otimes 1)(\beta(x \otimes y \otimes g_1 \wedge \cdots \wedge g_n))$$

$$= \sum_{k=1}^{m} \sum_{i,j=1}^{n} (-1)^{i+j} \langle y, g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge \hat{g_j} \wedge \cdots \wedge g_n \rangle$$

$$\cdot \langle e_k, g_j \rangle \varepsilon_k(x) \otimes g_j$$

while

$$\alpha \circ (\square \otimes 1)(x \otimes y \otimes g_1 \wedge \cdots \wedge g_n)$$
$$= \alpha \left(\sum_{k=1}^{n} \varepsilon_k(x) \otimes e_k \wedge y \otimes g_1 \wedge \cdots \wedge g_n \right)$$
$$= \sum_{k=1}^{m} \sum_{i=1}^{n} (-1)^i \langle e_k \wedge y, g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge g_n \rangle \varepsilon_k(x) \otimes g_i$$

and these are equal by the Laplace expansion formula.

Next we recall that $L_{(m-n+1)\times 2}M$ is the cokernel of the map $L_{(n,n-1)}F \otimes G \to^{\partial_\phi} L_{(n-1)\times 2}F$, or, equivalently, the cokernel of the map γ' which is the composition

$$\Lambda^n F \otimes \Lambda^{n-1} F \otimes G \xrightarrow{\gamma} \Lambda^{n-1} F \otimes \Lambda^{n-1} F \xrightarrow{\psi} L_{(n-1)\times 2}F,$$

where $\gamma(x \otimes y \otimes z) = \sum_{i=1}^{m} \langle e_i, z \rangle \varepsilon_i(x) \otimes y$. So in order to show that $\bar{\alpha}'$ factors as shown in $(**)$

$$\begin{array}{ccc} L_{(n-1)\times 2}F \otimes \Lambda^n G & \longrightarrow & L_{(m-n+1)\times 2}M \otimes \Lambda^n G \\ & {\scriptstyle \bar{\alpha}'} \searrow \quad \swarrow {\scriptstyle \bar{\alpha}''} & \\ & \Lambda^{m-n+1}M \otimes G & \end{array} \qquad (**)$$

all we have to show is $\bar{\alpha}' \circ \gamma' = 0$. Observe that showing $(**)$ concludes the proof because $\mathrm{Im}(\bar{\alpha}'') = \mathrm{Im}(\bar{\alpha}') = \mathrm{Im}(\bar{\alpha})$ is contained in $H_1(\Lambda^{m-n+2}\phi)$ by part (c).

It is trivial to check that there is a commutative diagram

$$\begin{array}{ccc} \Lambda^n F \otimes \Lambda^{n-1} F \otimes G \otimes \Lambda^n G & \xrightarrow{\gamma \otimes 1} & \Lambda^{n-1} F \otimes \Lambda^{n-1} F \otimes G \\ \eta \downarrow & & \downarrow \alpha \\ \Lambda^n F \otimes G \otimes G & \xrightarrow{\partial_\phi \otimes 1} & \Lambda^{n-1} F \otimes G \end{array}$$

where $\eta(x \otimes y \otimes z \otimes g_1 \wedge \cdots \wedge g_n) = \sum_{i=1}^{n} (-1)^i \langle y, g_1 \wedge \cdots \wedge \hat{g_i} \wedge \cdots \wedge g_n \rangle$ $x \otimes z \otimes g_i$. From this it follows that $\bar{\alpha} \circ (\gamma \otimes 1) = (\pi \otimes 1) \circ \alpha \circ (\gamma \otimes 1) = (\pi \otimes 1) \circ (\partial_\phi \otimes 1) \circ \eta = 0$ because $\pi \circ \partial_\phi = 0$. Therefore, $\bar{\alpha}' \circ (\gamma' \otimes 1) = \bar{\alpha}' \circ (\psi \otimes 1) \circ (\gamma \otimes 1) = \bar{\alpha} \circ (\gamma \otimes 1) = 0$ as desired.

Since $H_i(\Lambda^{m-n+2}\phi) = 0$ for $i \geq 2$, the comparison theorem enables us to lift the map $\alpha_*: L_{(n-1)\times 2}M \otimes \Lambda^n G \to H_1(\Lambda^{m-n+2}\phi)$ to a map $\boldsymbol{\alpha} = \{\alpha_k\}$: $L_{(m-n+1)\times 2}(\phi) \otimes \Lambda^n G \to \Lambda^{m-n+2}\phi$ of complexes as indicated below:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & L_{(n,n-1)}F \otimes G \otimes \Lambda^n G & \longrightarrow & L_{(n-1)\times 2}F \otimes \Lambda^n G & \longrightarrow & 0 \\
& \big\downarrow{\scriptstyle \alpha_1} & & \big\downarrow{\scriptstyle \alpha_0} & & \big\downarrow \\
\cdots \longrightarrow & \Lambda^n F \otimes D_2 G & \longrightarrow & \Lambda^{n-1}F \otimes G & \longrightarrow & \Lambda^{n-2}F.
\end{array}$$

(See a similar argument in the proof of Proposition 5.8 in [2].) Observe that each α_k is homogeneous of degree $n-1$.

We define the complex $\mathbb{X}(\boldsymbol{\alpha})$ to be the mapping cone of the map α, indexed so that the module $X_0(\boldsymbol{\alpha})$ of 0-chains is $\Lambda^{n-2}F$. The complex $\mathbb{X}(\boldsymbol{\alpha})$ is uniquely determined, up to homotopy, by the map ϕ.

THEOREM V.2.5. *$\mathbb{X}(\boldsymbol{\alpha})$ is a finite free resolution of the module $\Lambda^{m-n+2}M$. When $n \geqslant 2$, this is a minimal free resolution.*

Proof. We already know $H_0(\mathbb{X}(\boldsymbol{\alpha})) = H_0(\Lambda^{m-n+2}\phi) = \Lambda^{m-n+2}M$ by V.2.2, so all we have to show is that $\mathbb{X}(\boldsymbol{\alpha})$ is acyclic. Since $L_{(m-n+1)\times 2}\phi$ is acyclic (V.1.17) and $H_i(\Lambda^{m-n+2}\phi) = 0$ for $i \geqslant 2$, this is equivalent to $\alpha_*: H_0(L_{(m-n+1)\times 2}\phi) \to H_1(\Lambda^{m-n+2}\phi)$ being an isomorphism. Using the acyclicity lemma we will reduce the problem to the case $n = 1$.

If $n \geqslant 2$ then the length of $\mathbb{X}(\boldsymbol{\alpha})$ is $2(m-n+1)+2 = 2(m-n+2)$. By the acyclicity lemma it is enough to check that $\mathbb{X}(\alpha)$ is acyclic after localizing at primes P such that $\operatorname{grade}(PR_P) < 2(m-n+2)$. Since the grade of $I_{n-1}(\phi)$ is $2(m-n+2)$ we must have $I_{n-1}(\phi_P) = R_P$ for such a prime P. This means that an $(n-1)\times(n-1)$ minor of ϕ_P is a unit in R_P so that it is sufficient to prove the acyclicity of $\mathbb{X}(\alpha)$ after inverting an $(n-1)\times(n-1)$ minor of ϕ. In this case ϕ can be put in the form $1 \oplus \phi^1: R^{n-1} \oplus G^1 \to R^{n-1} \oplus F^{1*}$, where $\operatorname{rank}(G^1) = 1$, $\operatorname{rank}(F^1) = m-n+1$, and $\phi: G^1 \to F^{1*}$ is a generic map over R. By V.1.16 the complexes $L_{(m-n+1)\times 2}(\phi)$ and $\Lambda^{m-n+2}(\phi)$ are homotopically equivalent to $L_{(m-n+1)\times 2}(\phi^1)$ and $\Lambda^{m-n+2}(\phi^1)$. Using the definition of α_* given in Proposition V.2.4 it is easy to see that $\mathbb{X}(\boldsymbol{\alpha})$ is homotopically equivalent to the complex $\mathbb{X}(\boldsymbol{\alpha}^1)$ associated to the map ϕ^1. We have therefore reduced to the case $n = 1$.

When $n = \operatorname{rank}(G) = 1$, ϕ is a generic map $R \to F^*$. In this case $L_{m\times 2}\phi$ and $\Lambda^{m-1}\phi$ are both isomorphic to the Koszul complex associated to the generic map $\phi^*: F \to R$. The map α, as defined in V.2.4, is the identity $R \to R$ and therefore the induced map $\alpha_*: R/I \to R/I$, where $I = I_1(\phi)$, is also the identity. This proves the acyclicity of $\mathbb{X}(\boldsymbol{\alpha})$.

As for the minimality when $n \geqslant 2$, recall that the maps α_k are maps of degree $n-1 \geqslant 1$ over the graded ring $R = K[X_\phi]$ and that the boundary maps of the Schur complexes $L_{(m-n+1)\times 2}\phi$ and $\Lambda^{m-n+2}\phi$ are of degree one. Therefore the coefficients of the boundary maps of $\mathbb{X}(\alpha)$ are in the ideal $I_1(\phi)$. Since $I_1(\phi)$ is the homogeneous ideal generated by the elements of positive degree in $K[X_\phi]$ this is precisely what is meant by the minimality of the complex $\mathbb{X}(\alpha)$.

APPENDIX

Let K be a field of characteristic zero. The algebraic group $GL(n, K)$ is linearly reductive, which means that every polynomial representation of $GL(n, K)$ is completely reducible. We will briefly describe the classical definition of Schur functors, which give the irreducible polynomial representations of $GL(n, K)$ (for complete treatments, see [7, 9, 16, 21]).

If $\lambda = (\lambda_1, ..., \lambda_q)$ is a partition of weight α, we define a tableau T_λ in $\mathrm{Tab}_\lambda(\{1, ..., \alpha\})$ by $T_\lambda(i, j) = \lambda_1 + \cdots + \lambda_{i-1} + j$. For example,

$$T_{(4,3,1)} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

Let a_λ, b_λ be the elements in the group algebra $K[\mathrm{Sym}(d)]$ defined by $a_\lambda = \sum (-1)^\sigma \sigma$, $b_\lambda = \sum \tau$, where σ runs over all permutations in $\mathrm{Sym}(d)$ which preserve the rows of T_λ and τ runs over all those which preserve the columns of T_λ. Let c_λ be the Young symmetrizer $b_\lambda a_\lambda$ which is a primitive (pseudo-)idempotent of the group algebra $K[\mathrm{Sym}(d)]$.

Let V be a K-vector space of rank n. The symmetric group $\mathrm{Sym}(d)$ acts on the d-fold tensor product $\otimes^d(V) = V \otimes \cdots \otimes V$ by $\sigma(v_1 \otimes \cdots \otimes v_d) = v_{\sigma^{-1}_{(1)}} \otimes \cdots \otimes v_{\sigma^{-1}_{(d)}}$. The Schur functor $L_\lambda(V)$ is the subspace of $\otimes^d(V)$ defined by $c_\lambda \cdot \otimes^d(V)$. For example, if $\lambda = (d)$, then $c_\lambda = a_\lambda$ and $a_\lambda \cdot \otimes^d(V)$ is the subspace of antisymmetric tensors, $\Lambda^d(V)$. Similarly, if $\lambda = (d)^\sim = (1, ..., 1)$, then $c_\lambda = b_\lambda$ and $b_\lambda \cdot \otimes^d(V)$ is the subspace of symmetric tensors, $S_d(V)$.

The spaces $\{L_\lambda(V) \mid |\lambda| = d,\ \lambda_1 \leqslant n\}$ form a complete set of distinct irreducible polynomial representations of $GL(V)$ of degree d. It should be noted that $a_\lambda b_\lambda \cdot \otimes^d(V)$ is the coSchur functor $K_{\tilde{\lambda}}(V)$.

Since the category $M_K(n)$ of polynomial representations of $GL(n, K)$ is semisimple, its structure is completely determined by its Grothendieck ring, which is isomorphic, as a λ-ring, to the ring of symmetric polynomials in n-variables [11, 13]. It is often convenient to consider all n simultaneously, in which case the above statement becomes: the Grothendieck ring of the (semisimple) category $M_K(\infty)$ of polynomial functors is isomorphic to the ring of symmetric functions in an infinite number of variables [17].

When K is an infinite field of positive characteristic, the general linear group is no longer linearly reductive. Moreover, the definition of $L_\lambda(V)$ given above must be replaced by a universal definition, such as the one given in Chapter II. There $L_\lambda(V)$ is defined as the image of a natural transformation $d_\lambda(V)$ from $\Lambda^{\lambda_1} V \otimes \cdots \otimes \Lambda^{\lambda_q} V$ to $S_{\tilde{\lambda}_1} V \otimes \cdots \otimes S_{\tilde{\lambda}_t} V$. The exterior powers $\Lambda^d V$ are the only Schur functors which are irreducible in every characteristic. However, the Schur functors $L_\lambda V$ are always indecomposable (as

$GL(V)$-modules). This is a consequence of the following useful theorem which can be found in [8, Theorem 3.3].

THEOREM. *Let K be any infinite field and let $e_1, \ldots, e_n$ denote the canonical basis of K^n. Define $C_\lambda \in L_\lambda(K^n)$ to be $d_\lambda(e_1 \wedge \cdots \wedge e_{\lambda_1} \otimes \cdots \otimes e_1 \wedge \cdots \wedge e_{\lambda_q})$. Let $U^-(n, K) \subseteq GL(n, K)$ be the subgroup of lower triangular matrices with ones on the diagonal. Then: (1) Every $U^-(n, K)$-submodule of $L_\lambda(K^n)$ contains C_λ; (2) the $U^-(n, K)$-invariants of $L_\lambda(K^n)$ are spanned (over K) by C_λ; (3) $L_\lambda(K^n)$ is $U^-(n, K)$-indecomposable.*

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