# The equations of conjugacy classes of nilpotent matrices

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# The equations of conjugacy classes of nilpotent matrices

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## Section 1. Introduction

Let $X$ be the set of $n \times n$ matrices over a field $k$ of characteristic 0. For a partition $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s)$ of $n$ we denote by $O(\mathbf{u})$ the set of nilpotent matrices in $X$ with Jordan blocks of sizes $\mathbf{u}_1, \ldots, \mathbf{u}_s$. We are interested in equations of the closure of $O(\mathbf{u})$ in $X$ i.e. in the generators of the ideal of polynomial functions on $X$ vanishing on $O(\mathbf{u})$. For $\mathbf{u} = (n)$, $O(\mathbf{u})$ is the set of all nilpotent matrices and an old result of Kostant proved in the fundamental paper [K] says that the equations are the $GL(n)$ – invariants in the coordinate ring of $X$ ($GL(n)$ acts on $X$ by conjugation). The problem of calculating the equations of $O(\mathbf{u})$ in general was proposed by DeConcini and Procesi in [D-P] where the authors calculated the generators of ideals of schematic intersections $O(\mathbf{u}) \cap D$ ($D$ is the set of diagonal matrices). DeConcini and Procesi, Tanisaki [T] and Eisenbud and Saltman [E-S] proposed different sets of generators of the ideals of $O(\mathbf{u})$. It follows from our main result that all their conjectures are true. Moreover, we construct minimal sets of generators for the ideals of $O(\mathbf{u})$.

We also calculate the generators of ideals of "rank varieties" introduced by Eisenbud and Saltman in [E-S].

The method of proof comes from the techniques used to calculate the syzygies of determinantal varieties ([L], [J-P-W], [P-W]). There is a canonical desingularisation $Y_\mathbf{v}$ of $O(\mathbf{u})$ which is a complete intersection in $X \times GL(n)/P_\mathbf{v}$ for a suitable parabolic subgroup $P_\mathbf{v}$ of $GL(n)$ ($\mathbf{v}$ denotes the conjugate partition of $\mathbf{u}$). The syzygies of $\mathcal{O}_{Y_\mathbf{v}}$ over $\mathcal{O}_{X \times GL(n)/P_\mathbf{v}}$ are given by the Koszul complex on the cotangent bundle of $GL(n)/P_\mathbf{v}$. One uses Bott's Theorem to calculate higher direct images of the exterior powers of this bundle to find the generators of the ideals of $O(\mathbf{u})$.

The paper is organized as follows. Section 2 contains preliminaries concerning representation theory and the use of higher direct images to calculate syzygies. In Sect. 3 we use Bott's Theorem to construct complexes $M^\mathbf{v}(\cdot)$ giving (non-minimal) resolutions of the ideals $\mathfrak{J}_\mathbf{v}$ of $O(\mathbf{u})$. We also give a precise description of the 0-th

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and 1-st term in $M^{\mathbf{v}}$. Using this we recover the theorem of Kraft and Procesi [K-P] that the varieties $O(\mathbf{u})$ are normal. This proof of normality is shorter and conceptually simpler than the original one. In Sect. 4 we identify the generators of the ideals $\mathfrak{I}_{\mathbf{v}}$ as linear combinations of minors of various sizes, thus proving the conjecture of Tanisaki. Section 5 contains the description of minimal sets of generators of $\mathfrak{I}_{\mathbf{v}}$'s. Finally in Sect. 6 we apply our technique to obtain formulas for the decomposition of the coordinate ring of $O(\mathbf{u})$ into representations of $GL(n)$. This leads to nice inductive formulas for generalized exponents of Kostant. As a consequence we get also the inductive formulas for Kostka-Foulkes polynomials.

## Section 2. Preliminaries and notation

(2.0) *Notation.* We denote by $E$ our basic vector space of dimension $n$. The space $X$ is identified with $\mathrm{Hom}(E, E) = E^* \otimes E$, the space of endomorphisms $A: E \to E$. The coordinate ring of $X$ is the polynomial ring in the entries $A_{ij}$ $(1 \leqq i, j \leqq n)$. We identify this ring with the symmetric algebra $\mathscr{S}(E^* \otimes E)$. For a given basis $(e_1, \ldots, e_n)$ in $E$ the entry $A_{ij}$ becomes $e_j^* \otimes e_i$. We will use this identification freely throughout the paper.

A *partition* of the natural number $m$ is a non-increasing sequence $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s)$ such that $\mathbf{a}_1 + \ldots + \mathbf{a}_s = m$. We identify the partitions $(\mathbf{a}_1, \ldots, \mathbf{a}_s)$ and $(\mathbf{a}_1, \ldots, \mathbf{a}_s, 0)$. We write $|\mathbf{a}| = m$ if $\mathbf{a}$ is a partition of $m$.

The *conjugate partition* to $\mathbf{a}$ is the partition $\mathbf{a}^\sim = (\mathbf{a}_1^\sim, \ldots, \mathbf{a}_t^\sim)$ where

$$\mathbf{a}_j^\sim = {}^{\#}(i \mid \mathbf{a}_i \geqq j).$$

Let $\mathbf{u} = (\mathbf{u}_1, \ldots, \mathbf{u}_s)$ be a partition of $n$. $O(\mathbf{u})$ denotes the conjugacy class of nilpotent matrices with Jordan blocks of sizes $\mathbf{u}_1, \ldots, \mathbf{u}_s$. We denote by $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_t)$ the dual partition to $\mathbf{u}$. $\mathfrak{I}_{\mathbf{v}}$ is the ideal of elements in $\mathscr{S}(E^* \otimes E)$ vanishing on $O(\mathbf{u})$. We denote by $X_{\mathbf{v}}$ the Zariski closure of $O(\mathbf{u})$. One should note that if $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_t)$ then $O(\mathbf{u})$ consists of such $A: E \to E$ that $\dim \mathrm{Ker} A^i = \mathbf{v}_1 + \ldots + \mathbf{v}_i$.

In [E-S] the authors introduced a more general notion of rank varieties. Let $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_t)$ be the partition of a number $l$, $l \leqq n$. Then we consider the variety of all $A: E \to E$ such that $\dim \mathrm{Ker} A^i \geqq \mathbf{v}_1 + \ldots + \mathbf{v}_i$ for all $l$. We will denote this variety also by $X_{\mathbf{v}}$. When $l < n$ then $X_{\mathbf{v}}$ contains non-nilpotent endomorphisms. The methods of this paper are valid for those more general varieties. One should note that for an arbitrary endomorphism $A: E \to E$ there exists a partition $\mathbf{v}$ such that $\dim \mathrm{Ker} A^i = \mathbf{v}_1 + \ldots + \mathbf{v}_i$ for all $i$.

Let $I, J$ be two subsets of $\{1, 2, \ldots, n\}$ of cardinality $i$. We denote by $(I, J)$ the minor of $A$ with the rows from $I$ and columns from $J$.

The ring of $GL(n)$ invariants in $\mathscr{S}(E^* \otimes E)$ is described by Chevalley's Theorem. It says that this ring is the polynomial ring in $n$ variables $t_i$ $(1 \leqq i \leqq n)$ where $t_i$ is homogeneous of degree $i$. More precisely $t_i$ as a function on $X$ is the coefficient of $T^{n-i}$ in the characteristic polynomial $P_A(T)$. In terms of minors $t_i$ is the sum of principal $i \times i$ minors i.e.

$$t_i = \sum_{I \subset \{1, 2, \ldots, n\},\ \# I = i} (I, I).$$

In Sect. 4 we will use the natural maps

$$\Delta\colon \wedge^{a+b} E \to \wedge^a E \otimes \wedge^b E$$
$$m\colon \wedge^a E \otimes \wedge^b E \to \wedge^{a+b} E.$$

The map $m$ is the exterior multiplication, $\Delta$ is the dual of exterior multiplication on $E^*$ and is given by the following formula

$$\Delta(e_1 \wedge \ldots \wedge e_{a+b}) = \sum \operatorname{sgn}(I, I')\, e_I \otimes e_{I'}$$

where $\operatorname{sgn}(I, I')$ is the sign of the permutation ordering $(I, I')$, $I'$ denoting the complement of $I$.

We will call $m$ the multiplication map and $\Delta$ the diagonal map.

(2.1) *Geometric calculation of syzygies.* The basic theorem in our approach comes from Kempf's method [Ke]. The exact result we need is a version of Theorem from section 1 from [P-W].

**Theorem.** *Let $X' \subset X$ be a closed subvariety of the affine space $X = A^N$. Let $Y' \subset X \times V$ be a desingularisation of $X'$ ($V$ is some projective variety). We assume that the sheaves $\mathcal{O}_V(i)$ for $i \geqq 0$ have no higher cohomology. Let us assume that $Y'$ is a locally complete intersection in $X \times V$ and that there exists a vector bundle $\mathcal{N}$ on $X \times V$, induced from $V$, such that the locally free resolution of $\mathcal{O}_{Y'}$ over $\mathcal{O}_{X \times V}$ is given by the Koszul complex $\wedge^{\bullet}\mathcal{N}$. Let $\pi$ be the projection $\pi\colon X \times V \to X$.*

*Then there exists a sequence of free $\mathcal{O}_X$-modules*

$$\mathbb{F}\colon\ 0 \to F_N \to F_{N-1} \to \ldots \to F_1 \to F_0 \to F_{-1} \to \ldots$$

*where $F_i = \Sigma \mathcal{R}^j \pi_*(\wedge^{i+j}\mathcal{N})$ and such that the homology of $\mathbb{F}$ in positive degrees is zero and $H_{-i}(\mathbb{F}) = \mathcal{R}^i \pi'_* \mathcal{O}_{Y'}$. In particular if $F_i = 0$ for $i < 0$ then the normalisation $X'$ has rational singularities and if $F_0 = \mathcal{O}_X$ then $X'$ is normal.*

*Proof.* The only thing we have to do is to construct the complex $B^{\bullet\bullet}$ existence of which one assumes in Sect. 1 of [P-W]. We consider the dual complex $P^{*\bullet} = (\wedge^{\bullet}\mathcal{N})^*$. There exists a number $s > 0$ such that $P^{*\bullet}(s)$ consists of 0 regular sheaves in the sense of Quillen ([Q], §8). We can thus construct the double complex $B^{*\bullet\bullet}$ whose $t$-th column is a canonical resolution of $P^{*t}$. The term $B^{*it}$ is of the form $T_{it} \otimes \mathcal{O}_{X \times V}(-s-i)$. By dualizing we get the double complex $B^{\bullet\bullet}$ whose terms are $T_{it}^* \otimes \mathcal{O}_{X \times V}(s+i)$ so they are $\pi_*$ – acyclic by our assumption on $V$. Double complex $B^{\bullet\bullet}$ satisfies properties $(3, 3')$ of Theorem 1 in [P-W] because the horizontal maps cover the map of degree 1 in coordinate functions on $X$ so they can be chosen to be of degree 1. One should also mention that in the case of $\mathcal{R}^i \pi_* \mathcal{O}_{Y'} = 0$ for $i > 0$ (which we are interested in) another $B^{\bullet\bullet}$ is just $S \cdot (A^N \to \mathcal{Q})$ where $\mathcal{Q}$ is the quotient $A^N/\mathcal{N}$ and $S \cdot (A^N \to \mathcal{Q})$ denotes the symmetric power of the map $A^N \to \mathcal{Q}$ (compare [A-B-W]).

(2.2) *Flag varieties.* In our application of the above theorem, $V$ will be a flag variety, so we fix here some notation concerning flags. For a partition $\mathbf{v}$ of a number $l$, $l \leqq n$, we will denote by $G/P_{\mathbf{v}}$ the partial flag variety. Its typical point is a sequence of subspaces of $E$

$$0 \subset R_{\nu_1} \subset R_{\nu_1+\nu_2} \subset \ldots \subset R_{\nu_1+\nu_2+\ldots+\nu_t} \subset E.$$

The subscript of each subspace denotes its dimension. We will also use the symbol $R_i$ to denote the tautological subbundle of dimension $i$ on the flag variety. $Q_{n-i}$ denotes the corresponding factorbundle. Thus on a flag variety we have tautological exact sequences $0 \to R_i \to E \to Q_{n-i} \to 0$. $E$ denotes here the trivial induced bundle.

(2.3) *Representations of $G = GL(n)$.* Let $G = GL(n)$ be the general linear group. Let $a = (a_1, a_2, \ldots, a_n)$ be a dominant integral weight for $G$ (this means that $a_1 \geqq a_2 \geqq \ldots \geqq a_n$ and al $a_i$'s are integers). We denote by $S_{(a_1, a_2, \ldots, a_n)} E$ the irreducible representation of $G$ corresponding to $a$. Those representations can be expressed in terms of Schur functors.

$$S_a E = S_{a'} E \otimes (\wedge^n E)^a n$$

where $a'$ denotes the partition $(a_1 - a_n, \ldots, a_{n-1} - a_n, 0)$ and $S_{a'} E$ denotes the corresponding Schur functor.

The best reference for the representation theory is [Hu] and for the Schur functors the reader could consult [MD] (Sect. A5; the Schur functors are called there the irreducible polynomial functors) and [D-C]. If the sequence $(a_1, a_2, \ldots, a_n)$ has repetitions then we use exponential notation, for example if $a = (3, 3, 1, 1, 1, -1)$ then we write $a = (3^2, 1^3, -1)$.

Sometimes we will distinguish the positive and negative parts of $a$. If $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_r)$ and $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_u)$ are two partitions then we denote by $(\mathbf{s}, 0^c, \backslash \mathbf{t})$ the weight $(\mathbf{s}_1, \ldots, \mathbf{s}_r, 0^c, -\mathbf{t}_u, \ldots, -\mathbf{t}_1)$. Of course here $c + r + u = n$.

*Examples.* The symmetric power $S_j E$ becomes in our notation $S_{(n, 0^{n-1})} E$. The exterior power $\wedge^j E$ becomes $S_{(1^j, 0^{n-j})} E$, the exterior power $\wedge^j E^*$ becomes $S_{(0^{n-j}, (-1)^j)} E$.

We will always use the dominant weight notation for the indexing of Schur functors.

(2.4) **Bott's Theorem.** *For calculation of higher direct images we will use the following version of Bott's Theorem.*

**Theorem.** *Let $E$ be an n-dimensional vector bundle on ascheme $X$. Let $G/P_r(E)$ be the relative grassmannian of r-subbundles of $E$, $0 \to R_r \to E \to Q_{n-r} \to 0$ – the tautological sequence on $G/P_r(E)$. Let us denote by $\pi$ the natural projection $\pi$: $G/P_r(E) \to X$. Let $a = (a_1, a_2, \ldots, a_r)$ and $b = (b_1, b_2, \ldots, b_{n-r})$ be two dominant integral weights. Let $c = (b_1, \ldots, b_{n-r}, a_1, \ldots, a_r)$ and $\varrho = (n-1, \ldots, 2, 1, 0)$. We denote by $W$ the permutation group on n letters and for $w \in W$ $w \cdot c = w(c + \varrho) - \varrho$. Then either there exists $w \in W$, $w \neq \mathrm{id}$ such that $w \cdot c = c$, and then $\mathscr{R}^* \pi_* (S_a R \otimes S_b Q) = 0$, or there exists a unique $w$ such that $w \cdot c$ is dominant. We denote the weight $w \cdot c$ for this $w$ by $(b \mid a)$. In that case*

$$\mathscr{R}^{l(w)} \pi_* (S_a R \otimes S_b Q) = S_{(b \mid a)} E \quad \textit{and} \quad \mathscr{R}^i \pi_* (S_a R \otimes S_b Q) = 0 \quad \textit{for } i \neq l(w).$$

*Remark.* In [J-P-W] the theorem is formulated only for positive $a_i$, $b_j$ but the generalisation follows directly from the definitions.

(2.5) *Cauchy's Formulas.* In our proof we will use two important formulas that give the decomposition of the symmetric and exterior powers of $E\otimes F$ in terms of Schur functors.

$$S_m(E\otimes F)=\sum_{|\mathbf{a}|=m} S_{\mathbf{a}}E\otimes S_{\mathbf{a}}F,$$

$$\wedge^m(E\otimes F)=\sum_{|\mathbf{a}|=m} S_{\mathbf{a}}E\otimes S_{\mathbf{a}^\sim}F.$$

## Section 3. Complexes $M^{\mathbf{v}}(\cdot)$ and the description of low terms in the resolution of $\mathscr{S}(E^*\otimes E)/\mathfrak{J}_{\mathbf{v}}$

Our approach to the problem of finding generators of the ideals $\mathfrak{J}_{\mathbf{v}}$ is to use a desingularisation of $X_{\mathbf{v}}$. We consider the variety $Y_{\mathbf{v}}\subset X\times G/P_{\mathbf{v}}$

$$Y_{\mathbf{v}}=\{(A;R_{v_1},R_{v_1+v_2}\dots R_{v_1+v_2+\dots+v_t})\mid A\,R_{v_1+\dots+v_i}\subset R_{v_1+\dots+v_{i-1}}\ \text{ for all } i\}.$$

One can easily see that the image of $Y_{\mathbf{v}}$ under the projection on $X$ equals $X_{\mathbf{v}}$, and that the projection induces a birational isomorphism between $Y_{\mathbf{v}}$ and $X_{\mathbf{v}}$. The space $Y_{\mathbf{v}}$ can also be viewed as the total space of the cotangent bundle of $G/P_{\mathbf{v}}$. The structure sheaf of $Y_{\mathbf{v}}$ is the symmetric algebra on the tangent bundle of $G/P_{\mathbf{v}}$. We denote this bundle by $\mathscr{T}_{\mathbf{v}}$. The next step is to express $\mathscr{T}_{\mathbf{v}}$ in terms of tautological bundles on $G/P_{\mathbf{v}}$. To do that let us recall that there is a well – known equivalence of categories

$$[\text{homogeneous vector bundles on } G/P_{\mathbf{v}}]\xrightarrow{h}[P_{\mathbf{v}}\text{-modules}]$$

where $h(V)$ is the fibre at the identity. Under this equivalence the tangent space to $GL(E)$ becomes $E^*\otimes E$ and the tangent space to $P_{\mathbf{v}}$ becomes the parabolic Lie algebra $\mathfrak{p}_{\mathbf{v}}$. Now it is obvious looking at the roots belonging to $\mathfrak{p}_{\mathbf{v}}$ that $\mathscr{T}_{\mathbf{v}}$ has a composition series whose associated graded object equals

$$(3.1)\qquad \mathscr{T}'_{\mathbf{v}}=R^*_{v_1}\otimes(R_{v_1+v_2}/R_{v_1})\oplus\dots\oplus R^*_{v_1+\dots+v_{t-1}}\otimes(R_{v_1+\dots+v_t}/R_{v_1+\dots+v_{t-1}}).$$

We define the bundle $\mathscr{S}_{\mathbf{v}}$ by the exact sequence

$$(3.2)\qquad 0\to\mathscr{S}_{\mathbf{v}}\to E^*\otimes E\to\mathscr{T}_{\mathbf{v}}\to 0$$

and observe that $\mathscr{S}_{\mathbf{v}}$ has a composition series with associated graded object

$$(3.3)\qquad \mathscr{S}'_{\mathbf{v}}=E^*\otimes R_{v_1}\oplus Q^*_{v_2+\dots+v_t}\otimes(R_{v_1+v_2}/R_{v_1})\oplus\dots$$
$$\dots\oplus Q^*_{v_t}\otimes(E/R_{v_1+\dots+v_{t-1}}).$$

One should mention that in the case of rank varieties i.e. when $|\mathbf{v}|=l<n$, the corresponding $\mathscr{T}'$ and $\mathscr{S}'$ have a different expression. (3.1) must be modified by the additional summand $E^*\otimes(E/R_{v_1+\dots v_{t-1}})$ and $\mathscr{S}'_{\mathbf{v}}$ has the decomposition

$$(3.3')\qquad \mathscr{S}'_{\mathbf{v}}=E^*\otimes R_{v_1}\oplus Q^*_{n-v_1}\otimes(R_{v_1+v_2}/R_{v_1})\oplus\dots$$
$$\dots\oplus Q^*_{n-v_1-\dots-v_{t-1}}\otimes(R_{v_1+\dots+v_t}/R_{v_1+\dots+v_{t-1}}).$$

We have already said that the structure sheaf $\mathcal{O}_Y$ is equal to the symmetric algebra $\mathcal{S}(\mathcal{T}_{\mathbf{v}})$, so its locally free resolution over $\mathcal{O}_{X\times G/P}$ equals

(3.4) $$\wedge(\mathcal{S}_{\mathbf{v}})\to\mathcal{O}_Y\to O.$$

We want to apply Theorem (2.1). Instead of calculating higher direct images of $\wedge^i(\mathcal{S}_{\mathbf{v}})$ we calculate higher direct images of $\wedge^i(\mathcal{S}'_{\mathbf{v}})$. This gives us more terms but we know that they form a (non minimal) complex with the same cohomology as the complex we would get from higher direct images of $\wedge^i(\mathcal{S}_{\mathbf{v}})$. We define now the complexes $M^{\mathbf{v}}(\cdot)=M^{\mathbf{v}}(E,E^*,\cdot)$

(3.5) $$M^{\mathbf{v}}(j)=\bigoplus_i \mathcal{R}^i\pi_*(\wedge^{i+j}\mathcal{S}'_{\mathbf{v}}),$$

(3.6) $$M^{\mathbf{v}}(\cdot):\ \ldots\to M^{\mathbf{v}}(j)\to M^{\mathbf{v}}(j-1)\to\ldots.$$

The differentials in $M^{\mathbf{v}}(\cdot)$ are defined in such a way that $H_i(M^{\mathbf{v}}(\cdot))=\mathcal{R}^i\pi_*(\mathcal{O}_Y)$. By Cauchy's formula

(3.7) $$\wedge^j\mathcal{S}'_{\mathbf{v}}=\sum S_{\tilde{\mathbf{a}}_1}E^*\otimes S_{\mathbf{a}_1}R_{\mathbf{v}_1}\otimes\ldots\otimes S_{\tilde{\mathbf{a}}_t}Q^*_{\mathbf{v}_t}\otimes S_{\mathbf{a}_t}(E/R_{\mathbf{v}_1+\ldots+\mathbf{v}_{t-1}})$$

where we sum over all $t$-tuples of partitions $\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_t$ such that $|\mathbf{a}_1|+\ldots+|\mathbf{a}_t|=j$.

We notice that all our constructions can be made for relative flag manifolds i.e. for a vector bundle $\mathcal{E}$ over some base space $Z$ instead of a vector space $E$ (the case $\mathcal{E}=E$ corresponds to $Z$ being a point). Then the terms in (3.3) except the first one form the composition series of $\mathcal{S}'_{\mathbf{v}^\#}$ in the relative situation of $\mathcal{E}=Q_{|\mathbf{v}^\#|}$ where $\mathbf{v}^\#=(\mathbf{v}_2,\mathbf{v}_3,\ldots,\mathbf{v}_t)$. This proves the following lemma.

(3.8) **Lemma.** *Let* $\mathbf{v}=(\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_t)$ *and* $\mathbf{v}^\#=(\mathbf{v}_2,\ldots,\mathbf{v}_t)$. *Then the terms of* $M^{\mathbf{v}}(\cdot)$ *are the higher direct images* $\mathcal{R}^i\pi_*$ *of*

$$\sum S_{\tilde{\mathbf{a}}}E^*\otimes S_{\mathbf{a}}R\otimes M^{\mathbf{v}^\#}(Q,Q^*,\cdot)$$

*where* $R:=R_{\mathbf{v}_1}$, $Q:=Q_{n-\mathbf{v}_1}$ *and* $\pi$ *is the projection of* $X\times G/P_{\mathbf{v}}$ *onto* $X$.

Now we investigate the relation between the complexes $M^{\mathbf{v}}(E,E^*,\cdot)$ and $M^{\mathbf{v}^\#}(Q,Q^*,\cdot)$. The terms in $M^{\mathbf{v}^\#}(Q,Q^*,\cdot)$ are of the form $S_bQ$ where $b$ is a dominant integral weight with possibly negative indices. We look at the part of $M^{\mathbf{v}}(E,E^*,\cdot)$ coming from $S_bQ\otimes\wedge(E^*\otimes R)$. For our purposes it is convenient to denote

(3.9) $$b=(\mathbf{s},0^c,\backslash\mathbf{t})=(\mathbf{s}_1,\ldots,\mathbf{s}_u,0^c,-\mathbf{t}_v,\ldots,-\mathbf{t}_1)$$

where $\mathbf{s}$ and $\mathbf{t}$ are two partitions, $u+v+c=n$. We assume that $S_bQ$ appears in degree $i$ in $M^{\mathbf{v}^\#}(Q,Q^*,\cdot)$. We will denote the part of $M^{\mathbf{v}}(E,E^*,\cdot)$ coming from $S_bQ\otimes\wedge(E^*\otimes R)$ by $K_{(\mathbf{s},0^c,\backslash\mathbf{t})}(E,E^*)$. By Bott's theorem it equals

(3.10) $$\begin{aligned}K_{(\mathbf{s},0^c,\backslash\mathbf{t})}(E,E^*)&=\sum\mathcal{R}^j\pi_*(S_{\tilde{\mathbf{a}}}E^*\otimes S_{\mathbf{a}}R\otimes S_{(\mathbf{s},0^c,\backslash\mathbf{t})}Q)\\&=\sum S_{\tilde{\mathbf{a}}}E^*\otimes S_{(\mathbf{s},0^c,\backslash\mathbf{t}|\mathbf{a})}E[i+|\mathbf{a}|-\#(\mathbf{s},0^c,\backslash\mathbf{t}|\mathbf{a})]\end{aligned}$$

where the number in the square bracket denotes the degree of the corresponding element and $\#(\mathbf{s},0^c,\backslash\mathbf{t}|\mathbf{a})$ denotes the length of the permutation $w$ used in Bott's theorem (compare (2.4)).

Now we prove the key combinatorial lemma. The main idea is to show that the term of $M^{v^\#}(Q, Q^*, i)$ can give terms of $M^v(E, E^*, j)$ *only in degrees j bigger than or equal to i.*

(3.11) **Lemma.** *Let $O \to R \to E \to Q \to O$ be the tautological sequence on the grassmannian Grass $(r, n)$. Let us consider the bundle $S_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}Q$. We assume that $r = \dim R \geqq \mathbf{t}_1$. Then the terms of $K(\mathbf{s}, 0^c, \backslash \mathbf{t})(E, E^*)$ of lowest degree are*

$$\begin{array}{lccc} \to & S_{(\mathbf{t}, 1^{c+1})}E^* \otimes S_{(\mathbf{s}, 0^c, \backslash \mathbf{t} | \mathbf{t}_1^\sim + c+1, \mathbf{t}_2^\sim, \ldots)} E & \to & S_{\mathbf{t}}E^* \otimes S_{(\mathbf{s}, 0^c, \backslash \mathbf{t} | \mathbf{t}^\sim)} E \\ & \| & & \| \\ \to & S_{(\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_v, 1^{c+1})} E^* \otimes S_{(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_u, 1^{c+1})} E & \to & S_{\mathbf{t}}E^* \otimes S_{\mathbf{s}}E. \end{array}$$

*Moreover, if $S_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}Q$ occurs in $M^{v^\#}(Q, Q^*, i)$ then $S_{\mathbf{t}}E^* \otimes S_{\mathbf{s}}E$ also occurs in $M^v(E, E^*, i)$. For all $\mathbf{a}$'s leading to non zero terms in $K_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}(E, E^*)$ the weight $(\mathbf{s}, 0^c, \backslash \mathbf{t} | \mathbf{a})$ has nonnegative terms. Finally, if $\mathbf{t}_1 \leqq r$, then $K_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}(E, E^*)$ consists of terms $S_{\mathbf{w}}E \otimes S_{\mathbf{z}}E^*$ with $\mathbf{z}_1 \leqq r$.*

*Proof.* By Bott's Theorem (2.4) the partitions $\mathbf{a}$ leading to non zero terms in $K_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}(E, E^*)$ are the ones for which the sequence

$$\begin{aligned} z(\mathbf{a}) &= (\mathbf{s}, 0^c, \backslash \mathbf{t}, \mathbf{a}) + (n-1, n-2, \ldots, 2, 1) \\ &= (\mathbf{s}_1 + n - 1, \ldots, \mathbf{s}_u + n - u, r + v + c - 1, \ldots, r + v, r + v - 1 - \mathbf{t}_v, \ldots, r - \mathbf{t}_1, \\ &\quad \mathbf{a}_1 + r - 1, \ldots, \mathbf{a}_r) \end{aligned}$$

has no repetitions. The term corresponding to such $\mathbf{a}$ has a dominant weight that comes from reordering $z(\mathbf{a})$ into a strictly decreasing sequence, and substracting $\varrho = (n-1, \ldots, 2, 1)$ from it. This term occurs in the place $p(\mathbf{a}) = |\mathbf{a}| - l(w)$ where $l(w)$ is the number of exchanges needed to reorder $z(\mathbf{a})$.

First of all we notice that if $r \geqq \mathbf{t}_1$ then all numbers in $z(\mathbf{a})$ are positive. This shows that the weight $(\mathbf{s}, 0^c, \backslash \mathbf{t} | \mathbf{a})$ has nonnegative terms.

Next, it is easy to see that if $\mathbf{b} = (\mathbf{a}_1, \ldots, \mathbf{a}_s + j, \mathbf{a}_{s+1}, \ldots, \mathbf{a}_r)$ then $p(\mathbf{b}) > p(\mathbf{a})$. Indeed, when reordering $z(\mathbf{b})$ the number $\mathbf{a}_s + j + r - s$ can be exchanged with at most $j - 1$ additional places compared to $\mathbf{a}_s + r - s$ in $z(\mathbf{a})$. This will account for an increase of at most $j - 1$ in $l(w)$.

On the other hand if $c + v \geqq \mathbf{a}_1$ then $(r + v + c + 1, \ldots, r + v, r + v - 1 - \mathbf{t}_v, \ldots, r - \mathbf{t}_1, \mathbf{a}_1 + r - 1, \ldots, \mathbf{a}_r)$ are $r + v + c$ numbers belonging to $\{0, 1, \ldots, r + v + c - 1\}$. They can be distinct for a unique partition $\mathbf{a}$ and it is easy to see that $\mathbf{a} = \mathbf{t}^\sim$. It is also clear that this $\mathbf{a}$ is the smallest partition leading to a non zero term in $K_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}(E, E^*)$ and by the previous reasoning its $p(\mathbf{a})$ is the smallest, so it furnishes the lowest term of $K_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}(E, E^*)$. Finally it is easy to check that $p(\mathbf{t}^\sim) = 0$. Similarly one identifies a unique partition $\mathbf{a}'$ with $p(\mathbf{a}') = 1$.

To prove the last statement of (3.11) we recall that the typical term in $K_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}(E, E^*)$ is $S_{(\mathbf{s}, 0^c, \backslash \mathbf{t} | \mathbf{a})}E \otimes S_{\mathbf{a}^\sim} E^*$. Since $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_r)$, $\mathbf{a}_1^\sim \leqq r$. This concludes the proof of (3.11).

Now we can state the main result of this section.

(3.12) **Theorem.** *The* 0*-th and* 1*-st term of* $M^{\mathbf{v}}(E, E^*, \cdot)$ *consists of higher direct images of* $M^{\mathbf{v}^\#}(Q, Q^*, 0) \otimes \wedge (E^* \otimes R)$ *and of* $M^{\mathbf{v}^\#}(Q, Q^*, 1) \otimes \wedge (E^* \otimes R)$.

*Proof.* The statement (3.11) shows that the 0-th and 1-st term of $M^{\mathbf{v}}(E, E^*, \cdot)$ can come only from the 0-th and 1-st term of $M^{\mathbf{v}^\#}(Q, Q^*, \cdot)$ if we can assume that all terms of $M^{\mathbf{v}^\#}(Q, Q^*, \cdot)$ satisfy the condition of (3.11). To show it we prove by induction on the number of parts in $\mathbf{v}$ that all terms $S_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}E$ in $M^{\mathbf{v}}(E, E^*, \cdot)$ satisfy $\mathbf{t}_1 \leqq \mathbf{v}_1$. Indeed, let us assume that this is true for $M^{\mathbf{v}^\#}(Q, Q^*, \cdot)$ (i.e. all terms there are of the form $S_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}Q$ with $\mathbf{t}_1 \leqq \mathbf{v}_2$). But in (3.11) $r = \mathbf{v}_1 \geqq \mathbf{v}_2$ so all terms in $M^{\mathbf{v}^\#}(Q, Q^*, \cdot)$ satisfy the condition of the last statement of (3.11). Thus it remains to show that in $S_{\mathbf{w}}E \otimes S_{\mathbf{z}}E^*$ with $z_1 \leqq \mathbf{v}_1$ we can only get the representations $S_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}E$ with $\mathbf{t}_1 \leqq \mathbf{v}_1$. But this follows immediately from the Littlewood-Richardson rule ([MD], Sect. 1.9 and A.7).

(3.14) *Remark.* The proof of Theorem (3.12) shows that for any $m$ the 0-th, 1-st, ..., $m$-th terms of $M^{\mathbf{v}}(E, E^*, \cdot)$ come only from 0-th, 1-st, ..., $m$-th terms of $M^{\mathbf{v}^\#}(Q, Q^*, \cdot)$.

(3.15) **Corollary.** *The varieties* $X_{\mathbf{v}}$ *are normal and have rational singularities.*

*Proof.* The Theorem (3.12) shows that $M^{\mathbf{v}}(E, E^*, \cdot)$ has no terms in negative degrees and that $M^{\mathbf{v}}(E, E^*, 0) = \mathcal{O}_X$. The corollary now follows from Theorem (2.1).

To find the equations of the rank varieties $X_{\mathbf{v}}$ we need to analyse the term $M^{\mathbf{v}}(E, E^*, 1)$. The Theorem (3.12) and Lemma (3.11) tell us that for $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_t)$, $M^{\mathbf{v}}(E, E^*, 1)$ is constructed inductively from $M^{\mathbf{v}^\#}(Q, Q^*, \cdot)$ where $\mathbf{v}^\# = (\mathbf{v}_2, \ldots, \mathbf{v}_t)$. $M^{\mathbf{v}}(E, E^*, 1)$ consists of two parts:

1) $M^{\mathbf{v}^\#}(Q, Q^*, 1)$, decomposes into bundles $S_{(\mathbf{s}, 0^c, \backslash \mathbf{t})}Q$ and each such term gives us a term $S_{\mathbf{s}}E \otimes S_{\mathbf{t}}E^*$ in $M^{\mathbf{v}}(E, E^*, 1)$,

2) $M^{\mathbf{v}^\#}(Q, Q^*, 0)$ gives us $\wedge^{\mathbf{v}_2 + \cdots + \mathbf{v}_t + 1} E \otimes \wedge^{\mathbf{v}_2 + \cdots + \mathbf{v}_t + 1} E^*$ (corresponding to the vanishing of the minors of order $\mathbf{v}_2 + \ldots + \mathbf{v}_t + 1$) in $M^{\mathbf{v}}(E, E^*, 1)$.

It is easy to see using this procedure that the only representations of $E$ that can appear in $M^{\mathbf{v}}(E, E^*, 1)$ are $S_{(1^i, 0^{n-2i}, (-1)^i)}E$ for various $i$ and in various homogeneous degrees. We identify them as linear combinations of minors of various sizes in the next section.

## Section 4. Tanisaki generators of $\mathfrak{J}_{\mathbf{v}}$. Proof of the main theorem

In this section we investigate the linear combinations of minors in $\mathcal{S}(E \otimes E^*)$ vanishing on $X_{\mathbf{v}}$. This analysis is then used to prove that elements of this type generate the ideals $\mathfrak{J}_{\mathbf{v}}$.

Let us recall that by the Cauchy formula we have

$$S_m(E \otimes E^*) = \sum_{|\mathbf{a}| = m} S_{\mathbf{a}}E^* \otimes S_{\mathbf{a}}E.$$

We look at the $p \times p$ minors of the matrix A. They correspond to $\wedge^p E^* \otimes \wedge^p E$. Using the Littlewood-Richardson rule we find

$$\wedge^p E^* \otimes \wedge^p E \cong \bigoplus_{0 \leqq i \leqq \min(p, n-p)} S_{(1^i, 0^{n-2i}, (-1)^i)}E.$$

We will denote the copy of $S_{(1^i, 0^{n-2i}, (-1)^i)}E$ inside of $\wedge^p E^* \otimes \wedge^p E$ by $U_{i,p}$, so we get

$$\wedge^p E^* \otimes \wedge^p E = \sum_{0 \leqq i \leqq \min(p, n-p)} U_{i,p}.$$

Next, we denote by $V_{i,p}$ the subspace of $\wedge^p E^* \otimes \wedge^p E$ generated by $U_{0,p} \oplus U_{1,p} \oplus \ldots \oplus U_{i,p}$. The point of introducing $V_{i,p}$ is that the embedding of $V_{i,p}$ into $\wedge^p E^* \otimes \wedge^p E$ can be simply expressed in terms of minors. Indeed $V_{i,p}$ is isomorphic to $\wedge^i E^* \otimes \wedge^i E$ and the embedding is given by multiplying by the invariant $t_{p-i}$ (compare (2.0)). This means that if $e_1, \ldots, e_n$ is a basis in $E$, the element $e_{a_1} \wedge e_{a_2} \wedge \ldots \wedge e_{a_i} \otimes e^*_{b_1} \wedge e^*_{b_2} \wedge \ldots \wedge e^*_{b_i}$ in $V_{i,p}$ becomes the sum of minors

$$\sum_{|J|=p-i} (a_1, \ldots, a_i, J \mid b_1, \ldots, b_i, J).$$

First we determine which $V_{i,p}$ vanish on $X_\mathbf{v}$.

(4.1) **Lemma.** *Let* $\mathbf{v}$ *be a partition,* $\mathbf{u}$ *its conjugate. Then the elements of the space* $V_{i,p}$ *vanish on the variety* $X_\mathbf{v}$ *iff*

$$p > n - |\mathbf{u}| + \mathbf{u}_1 + \ldots + \mathbf{u}_i - i. \tag{4.2}$$

*Proof.* Let the condition (4.2) be satisfied. We will show that $V_{i,p}$ vanishes on $X_\mathbf{v}$. Indeed, the typical element of $V_{i,p}$ is

$$\sum_{|J|=p-i} (A, J \mid B, J)$$

for fixed subsets $A$, $B$, both of cardinality $i$. However all minors $(A, J \mid B, J)$ vanish on $X_\mathbf{v}$ because it is enough to check that they vanish on the elements of $X_\mathbf{v}$ in canonical Jordan form.

Now we prove that if condition (4.2) is not satisfied, then $V_{i,p}$ does not vanish on $X_\mathbf{v}$. Let us assume that $p \leqq n - |\mathbf{u}| + \mathbf{u}_1 + \ldots + \mathbf{u}_i - i$. Let us denote $n - |\mathbf{u}|$ by $y$. We choose a canonical Jordan form of a matrix from $X_\mathbf{v}$ such that the boxes corresponding to eigenvalue 0 are in the lower right corner. We also choose $j$ so $\mathbf{u}_j > 1$, $\mathbf{u}_{j+1} = 1$ (if $\mathbf{u}_i > 1$ then $j = i$). Now we choose

$$A = [y+1, y+\mathbf{u}_1+1, \ldots, y+\mathbf{u}_1+\ldots+\mathbf{u}_j+1].$$

We can also choose $w_1, \ldots, w_j$ in such a way that $1 \leqq w_m \leqq \mathbf{u}_m$, $w_1 + \ldots + w_j = p + i$. Let

$$B = [y+w_1, y+\mathbf{u}_1+w_2, \ldots, y+\mathbf{u}_1+\ldots+\mathbf{u}_{j-1}+w_j].$$

We consider the element (*) $\sum_{|J|=p-j} (A, J \mid B, J)$.

Clearly its value on the general matrix from $X_\mathbf{v}$ is not zero, because such a matrix is generic in the upper left $y \times y$ corner. Our element belongs to $V_{j,p}$ which is contained in $V_{i,p}$. This proves the lemma.

(4.3) *Remark.* Lemma 4.1 is due to Eisenbud and Saltman [E-S]. In case of nilpotent orbits it was proved by Tanisaki [T]. Their notation was different from ours. Let us mention that $\lambda^t_d$ in [E-S] is our $V_{n-t,d}$. In case $j = i$ in the proof of (4.1) we see that the element (*) we consider belongs to $U_{i,p}$. Thus in case $j = i$ we have that $U_{i,p}$ vanishes on $X_\mathbf{v}$ if and only if the condition (4.2) is satisfied. This is true in general, but the proof is much more complicated. We do not include it here because the proof of the main result is independent of this statement.

The representations $V_{i,p}$ do not give independent generators of $\mathfrak{I}_\mathbf{v}$.

(4.4) **Lemma.** *For $i \geqq 1$ $V_{i,p+1}$ is contained in the ideal generated by $V_{i,p}$.*

*Proof.* The lemma follows directly from the Laplace expansion.

Let us denote by $\mathscr{I}_\mathbf{v}$ the ideal generated by representations $V_{i,p}$ satisfying condition (4.2). Our goal is to show that in fact $\mathfrak{I}_\mathbf{v} = \mathscr{I}_\mathbf{v}$. Lemmas (4.1) and (4.4) show that $\mathscr{I}_\mathbf{v}$ is generated by the invariants $U_{0,n-|\mathbf{v}|+1}, \ldots, U_{0,n}$ and by the representations $U_{i,\mathbf{v}(i)} (1 \leqq i \leqq n)$ where $\mathbf{v}(i) = \mathbf{u}_1 + \ldots + \mathbf{u}_i - i + 1$ (let us notice that if $\mathbf{v}(i) > \min(\mathbf{v}(i), n - \mathbf{v}(i))$ then $U_{i,\mathbf{v}(i)}$ is zero).

There is a useful way to describe graphically the representations $U_{i,p}$. We do it for a concrete example from which the general case is obvious.

(4.5) *Example.* Let $n = 12$, $\mathbf{u} = (3,3,2,2,1,1)$. We draw the following diagram. For each pair $(i,p)$ corresponding to $U_{i,p}$ we draw an $X$ in the place $(i,p)$ if $U_{i,p}$ vanishes on $O(\mathbf{u})$ and 0 if $U_{i,p}$ does not vanish on $O(\mathbf{u})$. In fact, according to (4.2), the highest $X$ in the zeroth column occurs in the first row, the highest $X$ in the $i$-th column (for $i > 0$) occurs in the $\mathbf{v}(i)$-th row, where $\mathbf{v}(i) = \mathbf{u}_1 + \ldots + \mathbf{u}_i - i + 1$. For $\mathbf{u} = (3,3,2,2,1,1)$ we have $\mathbf{v}(1) = 3$, $\mathbf{v}(2) = 5$, $\mathbf{v}(3) = 6$, $\mathbf{v}(4) = \mathbf{v}(5) = \mathbf{v}(6) = 7$, so we get a diagram

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $p=0$ | 0 | | | | | | |
| $p=1$ | $X$ | 0 | | | | | |
| $p=2$ | $X$ | 0 | 0 | | | | |
| $p=3$ | $X$ | $X$ | 0 | 0 | | | |
| $p=4$ | $X$ | $X$ | 0 | 0 | 0 | | |
| $p=5$ | $X$ | $X$ | $X$ | 0 | 0 | 0 | |
| $p=6$ | $X$ | $X$ | $X$ | $X$ | 0 | 0 | 0 |
| $p=7$ | $X$ | $X$ | $X$ | $X$ | $X$ | $X$ | |
| $p=8$ | $X$ | $X$ | $X$ | $X$ | $X$ | | |
| $p=9$ | $X$ | $X$ | $X$ | $X$ | | | |
| $p=10$ | $X$ | $X$ | $X$ | | | | |
| $p=11$ | $X$ | $X$ | | | | | |
| $p=12$ | $X$ | | | | | | |
| $i:=$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

$\mathscr{I}_\mathbf{v}$ is generated by the first column and the highest $X$ in other columns, i.e. by $U_{0,p} (1 \leqq p \leqq 12)$, $U_{1,3}$, $U_{2,5}$, $U_{3,6}$, $U_{4,7}$, $U_{5,7}$.

Now comes the main result of this paper.

(4.6) **Theorem.** *For each partition $\mathbf{v}$ of $l$, $l \leqq n$, the ideal $\mathfrak{I}_\mathbf{v}$ of rank variety is equal to $\mathscr{I}_\mathbf{v}$ i.e. is generated by $U_{i,\mathbf{v}(i)} (1 \leqq i \leqq n)$, where as above $\mathbf{v}(i) = \mathbf{u}_1 + \ldots + \mathbf{u}_i - i + 1$, and by the invariants $U_{0,p} (n - |\mathbf{v}| + 1 \leqq p \leqq n)$.*

(4.7) *Remark.* This set of generators is not minimal. We will discuss minimal sets of generators of $\mathfrak{I}_\mathbf{v}$ in the next section.

*Proof of Theorem* (4.6). We argue by induction on the number $t$ of parts in $\mathbf{v}$. For $t=1$, $\mathbf{v}=(l)$ $X_{\mathbf{v}}$ is the set of matrices of rank $\leqq n-l$. The ideal $\mathfrak{I}_{\mathbf{v}}$ is generated by the $n-l+1$ minors. On the other hand the condition (4.2) reads $p>n-|\mathbf{u}|$ because $u=(l^l)$ and if $i>1$ then we are in the range $U_{i,p}=0$. Thus both conditions agree. Let us mention that in this case the complex $M^{\mathbf{v}}(E,E^*,\cdot)=\mathscr{R}^*\pi_*\wedge(R\otimes E^*)$ is the Lascoux resolution. Let $\mathbf{v}^{\#}=(\mathbf{v}_2,\ldots,\mathbf{v}_t)$. We know by induction that $\mathfrak{I}_{\mathbf{v}^{\#}}$ has the generators as indicated in the theorem. This means that we can assume (by cutting down $M^{\mathbf{v}^{\#}}(Q,Q^*,\cdot)$) that

$$M^{\mathbf{v}^{\#}}(Q,Q^*,1)=\sum_{1\leqq i\leqq|\mathbf{v}^{\#}|}U_{i,\mathbf{v}^{\#}(i)}(Q,Q^*)+\sum_{n'-|\mathbf{v}^{\#}|+1\leqq p\leqq n'}U_{0,p}(Q,Q^*)$$

where $n'=n-\mathbf{v}_1$. Now we apply Theorem (3.12). It follows that $M^{\mathbf{v}}(E,E^*,1)$ after cancelling some terms consists of the trivial representations in degrees $n'-|\mathbf{v}^{\#}|+1,\ldots,n'$ (corresponding to invariants), the terms

$\wedge^iE^*\otimes\wedge^iE=\mathscr{R}^i\pi_*(S_iR\otimes\wedge^iE^*\otimes S_{(1^i,0^{n'-2i},(-1)^i)}Q)$ in degree $\mathbf{v}^{\#}(i)+i$, and the term $\wedge^{\mathbf{v}_2+\ldots+\mathbf{v}_t+1}E^*\otimes\wedge^{\mathbf{v}_2+\ldots+\mathbf{v}_t+1}E$ corresponding to the rank condition on vanishing of minors of size $\mathbf{v}_2+\ldots+\mathbf{v}_t+1$ (coming from $M^{\mathbf{v}^{\#}}(Q,Q^*,0)$). It is enough to identify the second set of generators because identification of invariants is obvious.

First let us make some general comments about the connecting homomorphisms in the spectral sequence of the filtration on $\mathscr{S}_{\mathbf{v}}$ we analyze. The complex $M^{\mathbf{v}^{\#}}(Q,Q^*,\cdot)$ is a complex of $\mathscr{S}(Q\otimes Q^*)$ modules. It induces naturally a complex with the same terms over $\mathscr{S}(Q\otimes E^*)$. Thus the connecting homomorphisms lowering homological degree all come from the extensions induced from $0\to R\to E\to Q\to 0$.

The map in $M^{\mathbf{v}^{\#}}(Q,Q^*,\cdot)$ corresponding to $U_{i,\mathbf{v}^{\#}(i)}$ factors through the embedding

$$S_{(1^i,0^{n'-2i},(-1)^i)}Q\to\wedge^{\mathbf{v}^{\#}(1)}Q^*\otimes\wedge^{\mathbf{v}^{\#}(1)}Q.$$

Thus by the above remarks we see that the corresponding map in $M^{\mathbf{v}}(E,E^*,\cdot)$ factors as follows

$$\begin{array}{c}
\wedge^iE^*\otimes\wedge^iE=\mathscr{R}^i\pi_*(\wedge^iE^*\otimes S_iR\otimes S_{(1^i,0^{n'-2i},(-1)^i)}Q)\\
\downarrow\\
\mathscr{R}^i\pi_*(\wedge^iE^*\otimes S_iR\otimes\wedge^{\mathbf{v}^{\#}(i)}Q^*\otimes\wedge^{\mathbf{v}^{\#}(i)}Q)\\
\downarrow f\\
\mathscr{R}^0\pi_*(\wedge^iE^*\otimes\wedge^iQ\otimes\wedge^{\mathbf{v}^{\#}(i)}Q^*\otimes\wedge^{\mathbf{v}^{\#}(i)}Q)\\
\downarrow g\\
\mathscr{R}^0\pi_*(\wedge^iE^*\otimes\wedge^iQ\otimes\wedge^{\mathbf{v}^{\#}(i)}E^*\otimes\wedge^{\mathbf{v}^{\#}(i)}Q)\\
\downarrow h\\
\wedge^iE^*\otimes\wedge^iE\otimes\wedge^{\mathbf{v}^{\#}(i)}E^*\otimes\wedge^{\mathbf{v}^{\#}(i)}E\\
\downarrow u\\
\mathscr{S}(E^*\otimes E).
\end{array}\tag{4.8}$$

Here the map $f$ is induced by the action of the natural generator $\wedge^i(E \to Q)$ of

$$\mathrm{Ext}^i(\wedge^i Q, S_i R) = H^i(S_i R \otimes \wedge^i Q^*) = S_{(0,0,\ldots,0)} E = k .$$

$g$ is induced by the embedding $Q^* \to E^*$ and $h$ by the identification $\mathscr{R}^0\pi_*(\wedge^i Q) = \wedge^i E$. Let us also observe that $h$, $u$ are $GL(E^*) \times GL(E)$ invariant since they preserve homogenous degree in $E$ and $E^*$. The map $f$ has to go to $\mathscr{R}^0\pi_*(\wedge^i E^* \otimes \wedge^i Q \otimes \wedge^{\mathbf{v}^{\#}(i)} Q^* \otimes \wedge^{\mathbf{v}^{\#}(i)} Q)$ because for other partitions $\mathbf{a}$ of $i$, $\mathrm{Ext}^i(S_{\mathbf{a}} Q, S_i R) = H^i(S_i R \otimes S_{\mathbf{a}} Q^*) = 0$.

Let us consider the composition of the first four maps in (4.8). Since it goes from homogeneous degree $i$ in $E$ and $E^*$ to homogeneous degree $i + \mathbf{v}^{\#}(i)$, the extra $\mathbf{v}^{\#}(i)$ components have to come from a trace map $t_{\mathbf{a}}: k \to S_{\mathbf{a}} E \otimes S_{\mathbf{a}} E^*$ for some partition $\mathbf{a}$ of $\mathbf{v}^{\#}(i)$.

Now we have the commutative diagram

$$\begin{array}{ccc} H^i(S_i R \otimes \wedge^i Q^*) & \xrightarrow{\ v\ } & H^0(\wedge^i Q \otimes \wedge^i Q^*) \\ \| & & \uparrow {\scriptstyle H^0(t_i)} \\ k & \xrightarrow{\ \mathrm{id}\ } & H^0(S_{(0,0,\ldots,0)} Q) \end{array}$$

where $v$ is induced by the extension $\wedge^i(E \to Q)$. This shows that the components corresponding to $\wedge^i E$ in the term $\wedge^i E^* \otimes \wedge^i E \otimes \wedge^{\mathbf{v}^{\#}(i)} E^* \otimes \wedge^{\mathbf{v}^{\#}(i)} E$ of (4.8) come entirely from trace. On the other hand all the maps involved are equal to the identity on the components corresponding to $\wedge^i E^*$, so the components from $\wedge^{\mathbf{v}^{\#}(i)} E^*$ come entirely from trace. This shows that $t_{\mathbf{a}}$ above is in fact the exterior power $t_{\mathbf{v}^{\#}(i)}: k \to \wedge^{\mathbf{v}^{\#}(i)} E \otimes \wedge^{\mathbf{v}^{\#}(i)} E^*$. We can conclude that the whole composition 4.8 is of the following form

$$\wedge^i E^* \otimes \wedge^i E \xrightarrow{1 \otimes t_{\mathbf{v}^{\#}(i)}} \wedge^i E^* \otimes \wedge^i E \otimes \wedge^{\mathbf{v}^{\#}(i)} E^* \otimes \wedge^{\mathbf{v}^{\#}(i)} E \xrightarrow{\ w\ }$$
$$\to \wedge^i E^* \otimes \wedge^i E \otimes \wedge^{\mathbf{v}^{\#}(i)} E^* \otimes \wedge^{\mathbf{v}^{\#}(i)} E \to \mathscr{S}(E^* \otimes E)$$

where $w$ is the identity on the components involving $E^*$ and on the components involving $E$ it equals

$$w': \wedge^i E \otimes \wedge^{\mathbf{v}^{\#}(i)} E \xrightarrow{1 \otimes \Delta_{23}} \wedge^i E \otimes \wedge^{\mathbf{v}^{\#}(i) - i} E \otimes \wedge^i E \xrightarrow{m_{12} \otimes 1} \wedge^{\mathbf{v}^{\#}(i)} E \otimes \wedge^i E .$$

Here $\Delta$ and $m$ denote the diagonal and multiplication map (compare (2.0)). It is well known that $w'$ is a $GL(E)$ invariant isomorphism. It follows that the composition (4.8) is contained in the image of the map

$$\wedge^i E^* \otimes \wedge^i E \xrightarrow{1 \otimes t_{\mathbf{v}^{\#}(i)}} \wedge^i E^* \otimes \wedge^i E \, \wedge^{\mathbf{v}^{\#}(i)} E^* \otimes \wedge^{\mathbf{v}^{\#}(i)} E \xrightarrow{\ w''\ } \mathscr{S}(E^* \otimes E)$$

for some $GL(E^*) \times GL(E)$ invariant map $w''$. It follows from [D-E-P] that the map $w''$ is a linear combination of the maps $w_j (0 \leqq j \leqq i)$

$$w_j: \wedge^{\mathbf{v}^{\#}(i)} E^* \otimes \wedge^{\mathbf{v}^{\#}(i)} E \otimes \wedge^i E^* \otimes \wedge^i E \to$$
$$\to \wedge^{\mathbf{v}^{\#}(i)+j} E^* \otimes \wedge^{\mathbf{v}^{\#}(i)+j} E \otimes \wedge^{i-j} E^* \otimes \wedge^{i-j} E \xrightarrow{\ w_j''\ } \mathscr{S}(E^* \otimes E) .$$

Here the first map is a combination of diagonal and multiplication on both sides (like $w'$ above) and $w''_j$ is an embedding via products of minors of sizes $\mathbf{v}^{\#}(i)+j$ and $i-j$. To finish the proof of the theorem it is enough to prove that for $j<i$ $v_j := w''_j(1\otimes t_{\mathbf{v}^{\#}(i)})$ all belong to $\mathscr{I}_{\mathbf{v}}$. Indeed, $v_i$ is one of the generators of $\mathscr{I}_{\mathbf{v}}$ because $\mathbf{u}_j=\mathbf{u}^{\#}_j+1$, so $\mathbf{v}(i)=\mathbf{v}^{\#}(i)+i$. Now let us look at $v_{i-j}(j>0)$. The image of a typical element is a combination of products of minors of size $\mathbf{v}^{\#}(i)+i-j$ and $j$. But the minors of size $\mathbf{v}^{\#}(i)+i-j$ include the trace element $t_{\mathbf{v}^{\#}(i)}$. Each such combination is in $\mathscr{I}_{\mathbf{v}}$ because

$$i+\mathbf{v}^{\#}(i)=\mathbf{u}_1+\ldots+\mathbf{u}_i-i \quad \text{and} \quad \mathbf{u}_i=\mathbf{u}^{\#i}+1 \qquad \text{so}$$
$$i+\mathbf{v}^{\#}(i)-j\geqq \mathbf{u}_1+\ldots+\mathbf{u}_{i-j}-i+j$$

and condition (4.2) is satisfied. This finally proves that our generator is a combination of elements of $\mathfrak{I}_{\mathbf{v}}$.

(4.9) *Remark*. It seems that in fact the generator $U_{i,\mathbf{v}^{\#}(i)}(Q,Q^*)$ gives the representation $V_{i,\mathbf{v}^{\#}(i)+i}(E,E^*)$ but we proved a weaker statement above. It would be very interesting to identify this generator more precisely.

(4.10) *Example*. The combinatorics above is quite complicated so let us give a concrete example of the inductive step. We will show how one gets generators of $\mathfrak{I}_{(6,4,2)}$ from the generators of $\mathfrak{I}_{(4,2)}$. Let us draw the diagram (4.5) for the partition $(4,2)$

| | | | | |
|---|---|---|---|---|
| $p=0$ | 0 | | | |
| $p=1$ | $X$ | 0 | | |
| $p=2$ | $X$ | $X$ | 0 | |
| $p=3$ | $X$ | $X$ | $X$ | $X$ |
| $p=4$ | $X$ | $X$ | $X$ | |
| $p=5$ | $X$ | $X$ | | |
| $p=6$ | $X$ | | | |
| $i:=$ | 0 | 1 | 2 | 3 |

This means that $\mathfrak{I}_{(4,2)}$ is generated by the invariants, by $U_{1,2}$, $U_{2,3}$ and by $U_{3,3}$. After passing to $\mathfrak{I}_{(6,4,2)}$, terms corresponding to $U_{1,2}$, $U_{2,3}$ and $U_{3,3}$ give generators $V_{1,3}$, $V_{2,5}$ and $V_{3,6}$ respectively. This covers the generators $U_{1,3}$, $U_{2,5}$, $U_{3,6}$ of $\mathfrak{I}_{(6,4,2)}$. The generators $U_{4,7}$ and $U_{5,7}$ come from the term $K_{(0,0,0,0,0,0)}(Q,Q^*)$.

## Section 5. Minimal sets of generators of the ideals $\mathfrak{I}_{\mathbf{v}}$

In this section we describe plausible minimal sets of generators of the ideals $\mathfrak{I}_{\mathbf{v}}$. Theorem (4.6) states that $\mathfrak{I}_{\mathbf{v}}$ is generated by the representations $U_{0,p}(n-|\mathbf{v}|+1\leqq p\leqq n)$ and $U_{i,\mathbf{v}(i)}(1\leqq i\leqq n)$ where $\mathbf{v}(i)=n-|\mathbf{v}|+\mathbf{u}_1+\ldots+\mathbf{u}_i-i+1$. We give the main statement right away.

(5.1) **Conjecture.** *The ideal* $\mathfrak{I}_\nu$ *is minimally generated by the representations* $U_{0,p}(n-|\nu|+1\leqq p\leqq n-|\nu|+\mathbf{u}_1)$ and $U_{i,\nu(i)}$ *for which the following condition is satisfied*

(5.2) $$\nu(i)-1>i/j(\nu(j)-1) \text{ for all } 1\leqq j<i.$$

(5.3) *Remark.* The most suggestive interpretation of the condition (5.2) comes when using the diagram 4.5. Let us look at the position $(i,p)=(0,1)$. The symbol $X$ corresponding to $U_{i,\nu(i)}$ in the diagram is part of a minimal set of generators of $\mathfrak{I}_\nu$ when there are no $X$'es to the right of (or on) the segment joining $(i,\nu(i))$ with $(0,1)$.

(5.4) *Example.* $U_{2,5}$ is not among the minimal generators of $\mathfrak{I}_{(6,4,2)}$ because the point (1.3) belongs to the segment (0,1)–(2,5) (compare the diagram from (4.5).

In fact $\mathfrak{I}_{(6,4,2)}$ is minimally generated by $U_{0,1}$, $U_{0,2}$, $U_{0,3}$, $U_{1,3}$, $U_{3,6}$, $U_{4,7}$, $U_{5,7}$.

We close this section with some examples of ideals $\mathfrak{I}_\nu$ for special partitions. In all of these examples we assume $|\nu|=n$ i.e. we look at the nilpotent orbit.

(5.5) *Example.* $\nu=(m,1^{n-m})$. In this case $X_\nu$ is a so called rank variety. It consists of nilpotent matrices of rank $\leqq n-m$. Using the induction process from Sect. 3 it is possible to calculate the whole complex $M^\nu(\cdot)$. Indeed, for $\nu^\#=(1^{n-m})$, $M^{\nu^\#}(Q,Q^*,\cdot)$ is the Koszul complex on the invariants in degrees $1,2,\ldots,n-m$. Thus $M^\nu(\cdot)$ becomes $K_{(0,0,\ldots,0)}(E,E^*)$ tensored with this Koszul complex $K_{(0,0,\ldots,0)}(E,E^*)$ equals by definition $\mathscr{R}^*\pi_*(\wedge(R\otimes E^*))$ which is just the Lascoux complex for minors of order $n-m+1$. This shows that in this case $M^\nu(\cdot)$ is the minimal resolution of $\mathscr{S}(E^*\otimes E)/\mathfrak{I}_\nu$ and $X_\nu$ is a complete intersection cut out by $U_{0,1},\ldots,U_{0,n-m}$ in the determinantal variety of matrices of rank $\leqq n-m$. This result was obtained independently and by other methods in [E-S].

(5.6) *Example.* $\nu=(\nu_1,\nu_2)$. The construction from Sect. 3 shows that $\mathfrak{I}_\nu$ is generated by $U_{0,1}$, $U_{0,2}$, $U_{1,2}$ and the rank condition – vanishing of the minors of size $\nu_2+1$. However it is easy to see using Laplace expansions that the ideal generated by $V_{1,2}=U_{0,2}+U_{1,2}$ contains $V_{\nu_2,\nu_2+1}$. Thus the minimal set of generators consists in this case of $U_{0,1}$, $U_{0,2}$, $U_{1,2}$ and $U_{\nu_2+1,\nu_2+1}$. In the case when $\nu_1=\nu_2$ this last representation is zero. This example recovers the characteristic 0 case of the result of Strickland [S].

(5.7) *Example.* $\nu$ is a rectangular partition, i.e. $\nu=(r^s)$. Using Example 5.6 (case $\nu_1=\nu_2$) and the induction from Sect. 3 we easily find out that $\mathfrak{I}_\nu$ is generated by $U_{0,1},\ldots,U_{0,s}$ and $U_{1,s}$. Those generators are clearly minimal.

We see that for the partitions from Examples (5.5)–(5.7) the Conjecture (5.1) is satisfied.

## Section 6. Application to generalized exponents

The method of Sect. 3 applies to the problem of calculating the generalized exponents. Let us recall first the basic definitions.

We assume throughout this section that $\dim E = n$, $|\mathbf{v}| = n$. We consider $A_{\mathbf{v}} = \mathcal{S}(E^* \otimes E)/\mathfrak{J}_{\mathbf{v}}$ – the coordinate ring of $X_{\mathbf{v}}$. It was proven by Kostant in [K] that for each dominant integral weight $a = (a_1, \ldots, a_n)$ the representation $S_a E$ appears in $A_{\mathbf{v}}$ finitely many times. $A_{\mathbf{v}}$ is a graded ring, so we can define polynomials $P_{a,\mathbf{v}}(t) = \Sigma m_{a,\mathbf{v},i} t^i$ where $m_{a,\mathbf{v},i}$ is the multiplicity of $S_a E$ in the $i$-th graded component of $A_{\mathbf{v}}$. Kostant in [K] gave formulas for calculation of the coefficients $m_{a,\mathbf{v},i}$ in terms of $SL(2)$ – triples associated to $\mathbf{v}$ but they are not very explicit. Thus the problem is to find nice expressions for the polynomials $P_{a,\mathbf{v}}(t)$.

There has been a substantial ammount of work done in this direction. One should mention here the work of R. Stanley and R. Gupta [G], [St] and the unpublished note of D. Peterson [P].

Our approach allows us to develop inductive formulas for $P_{a,\mathbf{v}}(t)$ in terms of Bott's algorithm and the Littlewood-Richardson rule.

We will use the following notation. $P(\mathbf{v}, t)$ is the element of the polynomial ring in one variable $t$ over the representation ring of $GL(n)$ defined as

$$P(\mathbf{v}, t) = \Sigma [A_{\mathbf{v},i}]\, t^i \tag{6.1}$$

where $[A_{\mathbf{v},i}]$ denotes the class of the $i$-th graded component of $A_{\mathbf{v}}$. We have by definition

$$P(\mathbf{v}, t) = \Sigma P_{a,\mathbf{v}}(t)[S_a E], \tag{6.2}$$

Now it follows from (3.15) that

$$A_{\mathbf{v}} = \pi_*(\mathcal{S}(\mathcal{T}_{\mathbf{v}})), \qquad \mathcal{R}^i\pi_*(\mathcal{S}(\mathcal{T}_{\mathbf{v}})) = 0 \quad \text{for } i > 0. \tag{6.3}$$

Our method consists of calculating inductively $\Sigma(-1)^i \mathcal{R}^i\pi_*(\mathcal{S}(\mathcal{T}_{\mathbf{v}}'))$ using the grassmannian. Let us take $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_t)$ $\mathbf{v}^\wedge = (\mathbf{v}_1, \ldots, \mathbf{v}_{t-1})$. Let $n' = |\mathbf{v}^\wedge|$. We consider the grassmannian Grass $(n', E)$. We denote by $R$ the tautological subbundle, by $Q$ the corresponding factorbundle. Suppose we know the polynomials $P_{a,\mathbf{v}^\wedge}(t)$ for all weights $a = (a_1, \ldots, a_{n'})$. We consider now the exact sequence

$$0 \to \mathcal{T}_{\mathbf{v}^\wedge}(R) \to \mathcal{T}_{\mathbf{v}} \to R^* \otimes Q \to 0 \tag{6.4}$$

where $\mathcal{T}_{\mathbf{v}^\wedge}(R)$ denotes the bundle $\mathcal{T}_{\mathbf{v}^\wedge}$ defined in Sect. 3 calculated in the relative situation where $E$ becomes $R$. We get the following formula for $P(\mathbf{v}, t)$,

$$\begin{aligned} P(\mathbf{v}, t) &= \sum_{i,j \geqq 0} (-1)^i \mathcal{R}^i\pi_*(S_j \mathcal{T}_{\mathbf{v}})\, t^j \\ &= \sum_{i,j',j'' \geqq 0} (-1)^i \mathcal{R}^i\pi_*(S_{j'} \mathcal{T}_{\hat{\mathbf{v}}}(R) \otimes S_{j''}(R^* \otimes Q)\, t^j \\ &= \sum_{i \geqq 0} (-1)^i \mathcal{R}^i\pi_* \left( \sum_a P_{a,\hat{\mathbf{v}}}(t)\, S_a R \otimes \sum_{j'' \geqq 0} S_{j''}(R^* \otimes Q)\, t^{j''} \right) \\ &= \sum_{a,\mathbf{b}} P_{a,\hat{\mathbf{v}}}(t)\, t^{|\mathbf{b}|} \left( \sum_{i \geqq 0} (-1)^i R^i\pi_*(S_a R \otimes S_{\mathbf{b}} R^* \otimes S_{\mathbf{b}} Q) \right). \end{aligned}$$

Now we can state the main result of this section. To do that let us denote $r = |\mathbf{v}^\wedge|$, $q = n - r$. For a dominant integral weight $d = (d_1, \ldots, d_n)$ let $W(i, r, d)$ be the subset of the Weyl group $W$ of $GL(n)$ consisting of such elements $w$ that

$l(w)=i$, and for which $w\cdot d=(b_1,\ldots,b_n)$ has the property $b_1\geqq\ldots\geqq b_q$, $b_{q+1}\geqq\cdots\geqq b_n$.

(6.5) **Theorem.** *Let $d=(d_1,\ldots,d_n)$ be a dominant integral weight. Then*

$$(6.6)\quad P_{d,\mathbf{v}}(t)=\sum_{i\geqq 0}(-1)^i \sum_{a,(b_1,\ldots,b_q,c_1,\ldots,c_r)\in W(i,r,d)} (LR)_{a,-b;c}\,P_{a,\mathbf{v}^\wedge}(t)\,t^{|b|}$$

*where $(LR)_{a,-b;c}$ is the multiplicity of $S_cR$ in $S_aR\otimes S_bR^*$.*

*Proof.* The theorem follows from the last formula above after decomposing the tensor product in the bracket and applying Bott's theorem. One should note that the coefficient (LR) can be calculated from the Littlewood-Richardson rule.

(6.7) *Remark.* We can get a similar formula for $P_{d,\mathbf{v}}(t)$ in terms of $P_{a,\mathbf{v}^\#}(t)$ where $\mathbf{v}^\#=(\mathbf{v}_2,\ldots,\mathbf{v}_t)$ by using the other induction $\mathbf{v}\to\mathbf{v}^\#$ and the corresponding grassmannian.

(6.8) *Remark.* The method of Peterson [P] of generalized exponents (i.e. the case $\mathbf{v}=(1^n)$) consists in fact in calculating the higher direct images not for $\mathcal{T}_\mathbf{v}$ but for the associated graded bundle $\mathcal{T}''_\mathbf{v}$ which is a sum of one dimensional bundles (they correspond to positive roots). Our method in this case seems to be more economical.

(6.9) *Remark.* When this paper was in preparation *I* got from R. Stanley the preprint [M] where the authors prove similar result to Theorem (6.5) (Theorem (2.2)).

There is a very interesting special case of Theorem (6.5). We can consider the schematic intersections of the varieties $X_\mathbf{v}$ with the set $T$ of diagonal matrices. Geometrically each of them is a point. Let us denote by $B_\mathbf{v}$ the graded Artin algebra we get in this way ($B_\mathbf{v}=A_\mathbf{v}/J$ where $J$ is the ideal generated by $A_{i,j}$, $i\neq j$). There is a natural action of the Weyl group $W$ on the algebras $B_\mathbf{v}$. We can define the Poincare polynomials $K'_{\mathbf{v},\mathbf{w}}(t)$

$$K'_{\mathbf{v},\mathbf{w}}(t)=\Sigma m_{\mathbf{w},\mathbf{v},i}t^i$$

where $m_{\mathbf{w},\mathbf{v},i}$ is the multiplicity of the irreducible representation $S_\mathbf{w}$ of $W$ in the $i$-th graded component of $B_\mathbf{v}$. The polynomials $K'_{\mathbf{w},\mathbf{v}}(t)$ are called the Kostka-Foulkes polynomials (compare [MD]). It turns out that those polynomials are a special case of the polynomials $P_{a,\mathbf{v}}(t)$.

Let $a=(a_1,\ldots,a_n)$ be a dominant integral weight with $a_1+\ldots+a_n=0$ and $a_1=1$. To a we associate a partition $\mathbf{a}$ of $n$ by letting $\mathbf{a}_i=-a_{n+1-i}+1$. Now we can state the result.

(6.10) **Theorem.** $P_{a,\mathbf{v}}(t)=K'_{\mathbf{a},\mathbf{v}}(t)$.

*Proof.* The inductive formula one gets when using the filtration of $\mathcal{T}_\mathbf{v}$ by one dimensional bundles is the same as the definition of $K'_{\mathbf{a},\mathbf{v}}(t)$ given in MacDonald's book [MD] (Sect. III.6).

This proof was communicated to me by J. Klimek.

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**Note added in proof**

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